

THE BEAUVILLE-BOGOMOLOV CLASS AS A CHARACTERISTIC CLASS

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ABSTRACT. Let X be any compact Kähler manifold deformation equivalent to the Hilbert scheme of length n subschemes on a $K3$ surface, $n \geq 2$. For each point $x \in X$ we construct a rank $2n - 2$ reflexive coherent twisted sheaf E_x on X , locally free over $X \setminus \{x\}$, with the following properties.

- (1) E_x is ω -slope-stable with respect to some Kähler class ω on X .
- (2) Set $\kappa(E_x) := ch(E_x) \exp\left(\frac{-c_1(E_x)}{2n-2}\right)$. It is well defined, even though $c_1(E_x)$ and $ch(E_x)$ are not. The characteristic class $\kappa_i(E_x) \in H^{i,i}(X, \mathbb{Q})$ is monodromy-invariant, up to sign. Furthermore, $\kappa_i(E_x)$ can not be expressed in terms of classes of lower degree, if $2 \leq i \leq n/2$.
- (3) The Beauville-Bogomolov class is equal to $c_2(TX) + 2\kappa_2(E_x)$.

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1. INTRODUCTION

1.1. The main results. An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold X , such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form. An irreducible holomorphic symplectic manifold of real dimension $4n$ admits a Riemannian metric with holonomy $Sp(n)$ [Be]. Such a metric is called *hyperkähler*.

Let S be a smooth Kähler K3 surface and $S^{[n]}$ the Hilbert scheme of length n zero dimensional subschemes of S . $S^{[n]}$ is an irreducible holomorphic symplectic manifold [Be]. An irreducible holomorphic symplectic manifold X is said to be of $K3^{[n]}$ -type, if X is deformation equivalent to $S^{[n]}$, for a K3 surface S . The moduli space of manifolds of $K3^{[n]}$ -type is 21-dimensional, if $n \geq 2$. In particular, a generic manifold of $K3^{[n]}$ -type is not the Hilbert scheme of any K3 surface.

Definition 1.1. Let X be an irreducible holomorphic symplectic manifold. An automorphism g of the cohomology ring $H^*(X, \mathbb{Z})$ is called a *monodromy operator*, if there exists a family $\mathcal{X} \rightarrow B$ (which may depend on g) of irreducible holomorphic symplectic manifolds, having X as a fiber over a point $b_0 \in B$, and such that g belongs to the image of $\pi_1(B, b_0)$ under the monodromy representation. The *monodromy group* $Mon(X)$ of X is the subgroup of $GL(H^*(X, \mathbb{Z}))$ generated by all the monodromy operators.

Parallel transport of a class α in $H^{2i}(S^{[n]}, \mathbb{Q})$, which is $Mon(S^{[n]})$ -invariant, defines a class α_X in any X of $K3^{[n]}$ -type. More generally, if $\text{span}_{\mathbb{Q}}\{\alpha\}$ is a non-trivial $Mon(S^{[n]})$ -character, then we get a well defined unordered pair $\pm\alpha_X$ of a class and its negative. Such a class α_X is of Hodge type (i, i) , by Lemma 3.2.

We define in the proposition below certain classes $\kappa_i(X)$ in $H^{i,i}(X, \mathbb{Q})$, for i an integer in the range $1 \leq i \leq \frac{n+2}{2}$, and for any irreducible holomorphic symplectic manifold X of $K3^{[n]}$ -type, $n \geq 2$. Given a coherent \mathcal{O}_X -module E of rank $r > 0$, set

$$\kappa(E) := ch(E) \cup \exp(-c_1(E)/r),$$

and let $\kappa_i(E) \in H^{2i}(X, \mathbb{Q})$ be the summand of $\kappa(E)$ of degree $2i$. Let \mathcal{E} be the ideal sheaf of the universal subscheme in $S \times S^{[n]}$, f_i the projection from $S \times S^{[n]}$ onto the i -th factor, $i = 1, 2$, and I_Z the ideal sheaf of a length n subscheme $Z \subset S$. Let

$$(1.1) \quad E_Z := \mathcal{E}xt_{f_2}^1(f_1^*(I_Z), \mathcal{E})$$

be the relative extension sheaf over $S^{[n]}$. E_Z is a torsion free reflexive sheaf of rank $2n - 2$, which is locally free away from the point of $S^{[n]}$ corresponding to the ideal sheaf I_Z (Proposition 4.5). Set $\kappa_i(S^{[n]}) := \kappa_i(E_Z) \in H^{2i}(S^{[n]}, \mathbb{Q})$.

Proposition 1.2. (Proposition 3.1 and Lemma 3.2). *Let i be an integer in the range $2 \leq i \leq \frac{n+2}{2}$. The class $\kappa_i(S^{[n]})$ is monodromy invariant, for even i . The pair $\{\kappa_i(S^{[n]}), -\kappa_i(S^{[n]})\}$ is monodromy invariant, for odd i . Parallel transport of the pair $\{\kappa_i(S^{[n]}), -\kappa_i(S^{[n]})\}$ yields a well*

defined unordered pair $\{\kappa_i(X), -\kappa_i(X)\}$ of classes of type (i, i) on any irreducible holomorphic symplectic manifold X of $K3^{[n]}$ -type, $n \geq 2$.

The class $\kappa_i(X)$ is non-trivial; it can not be expressed as a polynomial in classes of degree less than $2i$, if $i \leq \frac{n}{2}$ ([Ma1], Lemma 10). In contrast, the odd Chern classes $c_{2k+1}(TX)$ vanish, since TX is a holomorphic symplectic vector bundle.

Twisted coherent sheaves and their characteristic classes are reviewed in section 2. For the uninitiated reader it would suffice at this point to note that the data of a locally free twisted sheaf is equivalent to that of a projective bundle. Furthermore, the definition of the characteristic classes $\kappa_i(E)$ above may be extended to define characteristic classes of both twisted sheaves and projective bundles.

Theorem 1.3. *Let X be any irreducible holomorphic symplectic manifold of $K3^{[n]}$ -type and i an integer in the range $2 \leq i \leq \frac{n+2}{2}$. The class $\kappa_i(X)$ is a characteristic class of a (possibly twisted) reflexive coherent sheaf E_x of rank $2n-2$ on X , which is locally free away from a single point x of X .*

Theorem 1.3 is proven below after Theorem 1.6. The cohomology group $H^2(X, \mathbb{Z})$, of an irreducible holomorphic symplectic manifold X , admits a canonical, symmetric, non-degenerate, and primitive bilinear pairing $q \in \text{Sym}^2 H^2(X, \mathbb{Z})^*$ [Be]. Theorem 1.3 yields an expression of the Beauville-Bogomolov pairing in terms of characteristic classes, for X of $K3^{[n]}$ -type, $n \geq 2$, by the following Lemma. The inverse of q is a class in $\text{Sym}^2 H^2(X, \mathbb{Q})$, and we denote by q^{-1} its image in $H^4(X, \mathbb{Q})$ as well.

Lemma 1.4. *The following equation holds in $H^4(X, \mathbb{Q})$, for any X of $K3^{[n]}$ -type, $n \geq 2$.*

$$(1.2) \quad q^{-1} = c_2(TX) + 2\kappa_2(X).$$

The dimension of the subspace¹ $\text{span}\{q^{-1}, c_2(TX), \kappa_2(X)\}$ is 2, for $n \geq 4$, and 1, for $n = 2, 3$.

The Lemma is proven in section 8. The main technical result of this paper is the following Theorem. Let n be an integer ≥ 2 . Set $r := 2n - 2$. Let S be a $K3$ surface admitting an ample line bundle H of degree $2r^2 + r$, which is not the tensor power of another ample line-bundle of lower degree. Let \mathcal{M} be the moduli space of Gieseker-Maruyama H -stable sheaves F of rank r with determinant line-bundle H and $\chi(F) = 2r$. Let π_{ij} be the projection from $\mathcal{M} \times S \times \mathcal{M}$ onto the product of the i -th and j -th factors.

Theorem 1.5. *There exists a $K3$ surface S with an ample line-bundle H of degree $2r^2 + r$ as above, such that the moduli space \mathcal{M} has the following properties.*

- (1) \mathcal{M} is smooth and projective of dimension $2n$.
- (2) (Proposition 4.5 and Theorem 7.10) There exists an untwisted universal sheaf \mathcal{E} over $S \times \mathcal{M}$. The relative extension sheaf

$$(1.3) \quad E := \text{Ext}_{\pi_{13}}^1(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})$$

¹ When $n = 3$, the relation $4q^{-1} = 3c_2(TM)$ holds as well. It follows from Chern numbers calculations, by comparing two formulas for the Euler characteristic $\chi(S^{[n]}, L)$ of a line bundle L on $S^{[n]}$. One as a binomial coefficient $\chi(S^{[n]}, L) = \binom{\frac{q(c_1(L), c_1(L))}{2} + n + 1}{n}$ [EGL], the other provided by Hirzebruch-Riemann-Roch.

- over $\mathcal{M} \times \mathcal{M}$ is reflexive, of rank $2n - 2$, locally free away from the diagonal, and $\mathcal{O}_{\mathcal{M}}(1) \boxtimes \mathcal{O}_{\mathcal{M}}(1)$ -slope-stable, for some² ample line-bundle $\mathcal{O}_{\mathcal{M}}(1)$ over \mathcal{M} .
- (3) (Proposition 4.2) Let i be an integer satisfying $4 \leq 2i \leq n + 2$. If i is even, then the class $\kappa_i(E)$ in $H^*(\mathcal{M} \times \mathcal{M}, \mathbb{Q})$ is invariant under the diagonal action of $\text{Mon}(\mathcal{M})$. If i is odd, then the pair $\{\kappa_i(E), -\kappa_i(E)\}$ is $\text{Mon}(\mathcal{M})$ -invariant.
- (4) [Y1] Let F be a stable sheaf over S with isomorphism class $[F] \in \mathcal{M}$. The restriction of the sheaf E to $\{[F]\} \times \mathcal{M}$ is isomorphic to the relative extension sheaf $E_F := \mathcal{E}xt_{f_2}^1(f_1^*F, \mathcal{E})$ over \mathcal{M} . Furthermore, the pair (\mathcal{M}, E_F) is deformation equivalent to the pair $(S^{[n]}, E_Z)$, given in equation (1.1). In particular, $\kappa_i(E_F)$ is equal to the class $\kappa_i(\mathcal{M})$ of Proposition 1.2.
- (5) (Theorem 7.10) The sheaf E_F in part 4 is $\mathcal{O}_{\mathcal{M}}(1)$ -slope-stable, for a generic such F .

Part (1) of Theorem 1.5 follows from results of Mukai [Mu1] and our choice of H in Theorem 7.10. Fix an integer $n \geq 2$ and a moduli space \mathcal{M} as in Theorem 1.5. Associated to the Kähler class $c_1(\mathcal{O}_{\mathcal{M}}(1))$ is a twistor deformation $\mathcal{X} \rightarrow \mathbb{P}^1$ of \mathcal{M} as an irreducible holomorphic symplectic manifold. A reflexive sheaf F over $\mathcal{M} \times \mathcal{M}$ is said to be *projectively-hyperholomorphic*, if $\mathcal{E}nd(F)$ extends as a reflexive sheaf \mathcal{A} of associative unital algebras over the fiber product $\mathcal{X} \times_{\mathbb{P}^1} \mathcal{X}$, flat over \mathbb{P}^1 , and \mathcal{A} is locally isomorphic to the endomorphism algebra $\mathcal{E}nd(\mathcal{F})$ of a reflexive coherent sheaf (Definition 6.6). One can relax the notion of a coherent sheaf to that of a twisted coherent sheaf (Definition 2.1). Then the extension \mathcal{A} is globally the sheaf $\mathcal{E}nd(\mathcal{F})$ of an extension of F to a twisted reflexive sheaf \mathcal{F} over $\mathcal{X} \times_{\mathbb{P}^1} \mathcal{X}$. The sheaf E in Theorem 1.5 is projectively-hyperholomorphic, by the stability result in part (2) of Theorem 1.5, the monodromy-invariance result in part (3) of Theorem 1.5, and a deep result of Verbitsky ([Ve4] and Corollary 6.10 below). We use Verbitsky's theory of hyperholomorphic sheaves in order to deform the pair (\mathcal{M}, E) to a pair (X, E') , for any irreducible holomorphic symplectic variety X of $K3^{[n]}$ -type. More precisely, we prove the following statement.

Theorem 1.6. (Theorem 7.11) *Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$ -type. There exist a reduced, connected, projective curve C of arithmetic genus 0, which may be reducible, a smooth and proper family $\mathcal{X} \rightarrow C$ of irreducible holomorphic symplectic manifolds, a torsion-free reflexive coherent twisted sheaf \mathcal{G} of $\mathcal{O}_{\mathcal{X} \times_C \mathcal{X}}$ -modules of rank $2n - 2$, flat over C , and points s, t , in C , with the following properties. The fiber X_s of \mathcal{X} over s is isomorphic to \mathcal{M} , the restriction G_s of \mathcal{G} to $X_s \times X_s$ is isomorphic to the sheaf E in Theorem 1.5, and the fiber X_t is isomorphic to X .*

A conjectural alternative approach to the construction of the deformation in the above Theorem, for X a sufficiently small deformation of \mathcal{M} , is sketched in [Ma5]. This alternative approach is more geometric, but leads only to local deformations.

Proof. (Of Theorem 1.3) The pairs $\pm\kappa_i(E_F)$ and $\pm\kappa_i(\mathcal{M})$ are equal, by part 4 of Theorem 1.5. The pairs (\mathcal{M}, E) and (X, G_t) are deformation equivalent, where G_t is the restriction of \mathcal{G} to the fiber $X_t \times X_t \cong X \times X$ over the point t of C in Theorem 1.6. Given a point $x \in X$, denote by E_x the restriction of G_t to $\{x\} \times X$. The pairs (\mathcal{M}, E_F) and (X, E_x) are then deformation equivalent. The characteristic class $\kappa_i(E)$ is defined in section 2 for any twisted sheaf E . Theorem 1.6 thus implies that the pair $\pm\kappa_i(E_x)$ is a parallel transport of the pair

²More information about $\mathcal{O}_{\mathcal{M}}(1)$ is provided in Theorem 7.10.

$\pm\kappa_i(E_F)$. We conclude that the pairs $\pm\kappa_i(E_x)$ and $\pm\kappa_i(X)$ are equal, by Proposition 1.2. This completes the proof of Theorem 1.3. \square

The fact that the deformations of the sheaf E in Theorem 1.6 are twisted is a blessing, rather than a nuisance. It is key to the proof of its slope-stability. This is due to the following general result.

Proposition 1.7. *(Proposition 7.8) Let (X, ω) be a compact Kähler manifold, $\theta \in H_{an}^2(X, \mathcal{O}_X^*)$ a class of order $r > 0$, and E a reflexive, rank r , θ -twisted sheaf. Then $\mathcal{E}nd(E)$ is ω -slope polystable.*

1.2. Two applications.

1.2.1. *The Lefschetz standard conjecture.* The Lefschetz standard conjecture is proven in [CM] for any projective irreducible holomorphic symplectic manifold X of $K3^{[n]}$ -type. The proof uses the algebraic classes $\kappa_i(E)$ of the rank $2n - 2$ twisted reflexive sheaf E over $X \times X$ constructed above in Theorem 1.6. As a consequence, two algebraic cycles on X are numerically equivalent, if and only if they represent the same class in $H^*(X, \mathbb{Q})$.

1.2.2. *Non-commutative deformations of the derived category of K3 surfaces.* Let S , \mathcal{M} , and \mathcal{E} , be as in Theorem 1.5. Let $D^b(S)$ be the bounded derived category of coherent sheaves on S . We expect that the hyperkähler deformations of \mathcal{M} , which do not come from deformations of S , nevertheless correspond to “deformations” of $D^b(S)$ as follows. Consider the exact functor $\Phi_{\mathcal{E}} : D^b(S) \rightarrow D^b(\mathcal{M})$ with the universal sheaf $\mathcal{E} \in D^b(S \times \mathcal{M})$ as its kernel. Set $\mathcal{E}_R := \mathcal{E}^{\vee}[2]$ and let $\Phi_{\mathcal{E}_R} : D^b(\mathcal{M}) \rightarrow D^b(S)$ be the exact functor with kernel \mathcal{E}_R . Then $\Phi_{\mathcal{E}_R}$ is the right adjoint of $\Phi_{\mathcal{E}}$ [Mu4]. Set

$$\Phi := \Phi_{\mathcal{E}} \circ \Phi_{\mathcal{E}_R} : D^b(\mathcal{M}) \rightarrow D^b(\mathcal{M}).$$

The kernel $\mathcal{F} \in D^b(\mathcal{M} \times \mathcal{M})$, of the endo-functor Φ , is the convolution of \mathcal{E} and \mathcal{E}_R . Let π_{ij} be the projection from $\mathcal{M} \times S \times \mathcal{M}$ onto the product of the i -th and j -th factors. Then $\mathcal{F} := R_{\pi_{13}} \left(\pi_{12}^* \mathcal{E}_R \overset{L}{\otimes} \pi_{23}^* \mathcal{E} \right) [2]$. Hence, \mathcal{F} fits in an exact triangle $E[-1] \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{\Delta} \rightarrow E$, where E is given in equation (1.3) and $\Delta \subset \mathcal{M} \times \mathcal{M}$ is the diagonal.

The sheaf E deforms to a twisted sheaf over the self product of every X of $K3^{[n]}$ -type, $n := \dim(\mathcal{M})/2$, by Theorem 1.6. The kernel \mathcal{F} of Φ similarly deforms to an object in the derived category of twisted sheaves over $X \times X$ [MS]. We get a deformation of the endo-functor Φ to endo-functors of bounded derived categories of twisted sheaves $D^b(X, \theta)$, where $\theta \in H^2(X, \mu_{2n-2})$ is a monodromy invariant class, up to sign, given in Lemma 7.2.

The functor $\Phi_{\mathcal{E}}$ and the endo-functor Φ are studied in a joint work with S. Mehrotra [MS]. It is shown that $\Phi_{\mathcal{E}}$ is faithful when $\mathcal{M} = S^{[n]}$ and it is expected to be faithful for any \mathcal{M} as above. Whenever $\Phi_{\mathcal{E}}$ is faithful, the category $D^b(S)$ can be reconstructed from the co-monad (Φ, ϵ, δ) , where $\epsilon : \Phi \rightarrow id$ is the co-unit for the adjoint pair $(\Phi_{\mathcal{E}}, \Phi_{\mathcal{E}_R})$, $\eta : id \rightarrow \Phi_{\mathcal{E}_R} \circ \Phi_{\mathcal{E}}$ the unit, and $\delta := \Phi_{\mathcal{E}} \eta \Phi_{\mathcal{E}_R} : \Phi \rightarrow \Phi^2$ the co-action. Associated to the co-monad (Φ, ϵ, δ) is the category $\mathcal{C}(\Phi, \epsilon, \delta)$ of co-algebras for the co-monad ([Mac], section VI). The reconstruction of $D^b(S)$ is an application of the Bar-Beck Theorem ([Mac], section VI.7, and [MS]), which states that the natural functor $D^b(S) \rightarrow \mathcal{C}(\Phi, \epsilon, \delta)$ is an equivalence, when $\Phi_{\mathcal{E}}$ is faithful. We expect the co-monad structure (Φ, ϵ, δ) to deform along every hyperkähler deformation of \mathcal{M} ,

including deformations along which the K3 surface S does not deform. Consequently, the category $\mathcal{C}(\Phi, \epsilon, \delta)$ would deform along every hyperkähler deformation of \mathcal{M} .

1.3. Notation. Let $f : X \rightarrow Y$ be a proper morphism of complex manifolds or smooth quasi-projective varieties. We denote by f_* the push-forward of coherent sheaves, as well as the Gysin homomorphism in singular cohomology, while $f_!$ is the Gysin homomorphism in K -theory (algebraic, holomorphic [OTT], or topological). We let $K_{top}X$ be the Grothendieck K -ring of equivalence classes of formal sums of topological vector bundles over X .

The pullback homomorphism is denoted by f^* for coherent sheaves and in singular cohomology, while $f^!$ is the pull back in K -theory. Given a class α in $H^{even}(X)$, we denote by α_i the graded summand in $H^{2i}(X)$.

Given a Čech 2-cocycle θ of \mathcal{O}_X^* on a complex variety X , we define the notion of a θ -twisted coherent sheaf in Definition 2.1. A (coherent) sheaf will always mean an *untwisted* (coherent) sheaf, unless we explicitly mention that it is twisted.

2. CHARACTERISTIC CLASSES OF PROJECTIVE BUNDLES AND TWISTED SHEAVES

Let Y be a topological space and y a class in the ring $K_{top}Y$ generated by classes of complex vector bundles over Y . Assume that the rank r of y is non-zero. Set

$$(2.1) \quad \kappa(y) := ch(y) \cup \exp(-c_1(y)/r),$$

and let $\kappa_i(y)$ be the summand of $\kappa(y)$ in $H^{2i}(Y, \mathbb{Q})$. In terms of the Chern roots y_j , we have $ch_i(E) = \sum_{j=1}^r \frac{y_j^i}{i!}$, $c_1(E) = \sum_{j=1}^r y_j$, and

$$\kappa_i(y) = \sum_{j=1}^r \frac{\left[y_j - \left(\frac{\sum_{k=1}^r y_k}{r} \right) \right]^i}{i!}.$$

The characteristic class κ is multiplicative, $\kappa(y_1 \otimes y_2) = \kappa(y_1) \cup \kappa(y_2)$, and $\kappa([L]) = 1$, for any line bundle L . Given a vector bundle E over Y , the equality $\kappa(E) = \kappa(E \otimes L)$ thus holds, for any line bundle L . Note the equalities

$$\begin{aligned} \kappa_i(y^\vee) &= (-1)^i \kappa_i(y), \\ \kappa(-y) &= -\kappa(y). \end{aligned}$$

2.1. Characteristic classes and Brauer classes of projective bundles. We define next the invariant $\kappa(\mathbb{P})$, for any holomorphic \mathbb{P}^{r-1} -bundle, $r \geq 1$, over a complex variety Y , endowed with the analytic topology. The definition is clear, if \mathbb{P} is the projectivization of a vector bundle E , since $\kappa(E)$ is independent of the choice of E . More generally, the Brauer class

$$\theta(\mathbb{P}) \in H_{an}^2(Y, \mathcal{O}_Y^*)$$

is the obstruction class to lifting \mathbb{P} to a holomorphic vector bundle. The Brauer class $\theta(\mathbb{P})$ is the image of the class $[\mathbb{P}] \in H_{an}^1(Y, PGL_r)$, under the connecting homomorphism of the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_Y^* \rightarrow GL_r(\mathcal{O}) \rightarrow PGL_r(\mathcal{O}) \rightarrow 0.$$

Consider the dual bundle $\pi : \mathbb{P}^* \rightarrow Y$. The pullback $\pi^*\mathbb{P}$ has a tautological hyperplane subbundle, hence a divisor, hence a holomorphic line-bundle $\mathcal{O}_{\pi^*\mathbb{P}}(1)$. The obstruction class $\theta(\mathbb{P})$ is

in the kernel of $\pi^* : H_{an}^2(Y, \mathcal{O}_Y^*) \rightarrow H_{an}^2(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}^*}^*)$ and the projective bundle $\pi^*\mathbb{P}$ over \mathbb{P}^* is the projectivization of some vector bundle \tilde{E} . The class $\kappa(\tilde{E})$ belongs to the image of the injective homomorphism $\pi^* : H^*(Y, \mathbb{Q}) \rightarrow H^*(\mathbb{P}^*, \mathbb{Q})$, since $\kappa(\tilde{E})$ restricts as r to each fiber of π . Define³

$$(2.2) \quad \kappa(\mathbb{P}) \in H^*(Y, \mathbb{Q})$$

as the unique class satisfying $\pi^*(\kappa(\mathbb{P})) = \kappa(\tilde{E})$.

The class $\theta(\mathbb{P})$ is determined by a topological class, which we now define. Let μ_r be the group of r -th roots of unity. Denote the corresponding local system by μ_r as well, and let $\iota : \mu_r \rightarrow \mathcal{O}^*$ be the inclusion. Let

$$(2.3) \quad \tilde{\theta} : H_{an}^1(Y, PGL_r(\mathcal{O})) \rightarrow H^2(Y, \mu_r)$$

be the connecting homomorphism of the short exact sequence

$$0 \rightarrow \mu_r \rightarrow SL_r(\mathcal{O}) \rightarrow PGL_r(\mathcal{O}) \rightarrow 0.$$

Then the following equality clearly holds.

$$(2.4) \quad \theta(\mathbb{P}) = \iota[\tilde{\theta}(\mathbb{P})].$$

When \mathbb{P} is the projectivization of a vector bundle V over Y , the following equality holds ([HSc], Lemma 2.5)

$$(2.5) \quad \tilde{\theta}(\mathbb{P}V) = \exp\left(\frac{-2\pi\sqrt{-1}}{r}c_1(V)\right).$$

2.2. Twisted sheaves.

Definition 2.1. Let Y be a scheme or a complex analytic space, $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ a covering, open in the complex or étale topology, and $\theta \in Z^2(\mathcal{U}, \mathcal{O}_Y^*)$ a Čech 2-cocycle. A θ -twisted sheaf consists of sheaves E_α of \mathcal{O}_{U_α} -modules over U_α , for all $\alpha \in I$, and isomorphisms $g_{\alpha\beta} : (E_\beta)|_{U_{\alpha\beta}} \rightarrow (E_\alpha)|_{U_{\alpha\beta}}$ satisfying the conditions:

- (1) $g_{\alpha\alpha} = id$,
- (2) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$,
- (3) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \theta_{\alpha\beta\gamma} \cdot id$.

The θ -twisted sheaf is *coherent*, if the E_α are.

The abelian categories of θ -twisted and θ' -twisted coherent sheaves are equivalent, if the cocycles θ and θ' represent the same cohomology class. The equivalence is not canonical, but the ambiguity is only up to tensorization by an untwisted line-bundle [Ca]. Thus, the classes κ_i of a twisted sheaf, defined below, are preserved under the equivalences. We will often abuse terminology and refer to a θ -twisted sheaf, where θ is a class in $H_{an}^2(Y, \mathcal{O}_Y^*)$, meaning the equivalence class of θ -twisted sheaves, for different choices of Čech cocycles θ' , representing the class θ .

Remark 2.2. Observe that the determinant $\det(E)$, of a θ -twisted coherent torsion free sheaf E of rank r , is a θ^r -twisted line-bundle. Thus, θ^r is a coboundary. Consequently, the order of the class $[\theta]$, of θ in $H_{an}^2(Y, \mathcal{O}_Y^*)$, divides the rank of every θ -twisted torsion free sheaf E .

³ The construction has an analogue for topological complex \mathbb{P}^{r-1} -bundles. Note that the topological analogue of $H_{an}^2(Y, \mathcal{O}_Y^*)$, for the sheaf of invertible continuous complex valued functions, is isomorphic to $H^3(Y, \mathbb{Z})$, via the connecting homomorphism of the exponential sequence.

Assume Y is a complex manifold. A projective \mathbb{P}^{r-1} bundle \mathbb{P} over Y corresponds to a rank r locally-free twisted coherent sheaf E , with twisting cocycle θ in $Z^2(\mathcal{U}, \mathcal{O}_Y^*)$, for some open covering \mathcal{U} of Y . The θ -twisted sheaf E is unique, up to tensorization by a line-bundle. The characteristic class $\kappa(E) := \kappa(\mathbb{P})$ can be generalized for twisted sheaves, which are not locally free, as we show next.

Given θ_j -twisted sheaves E_j , $1 \leq j \leq k$, let $\mathcal{T}or_i^Y(E_1, \dots, E_k)$ be the i -th multi-Tor twisted sheaf. Over an open analytic subset $U \subset Y$, where each sheaf E_j admits a locally free resolution $(V_\bullet^j) \rightarrow E_j$, $(V_\bullet^j) := V_d^j \rightarrow \dots \rightarrow V_1^j \rightarrow V_0^j$, the sheaf $\mathcal{T}or_i^Y(E_1, \dots, E_k)$ is the i -th homology of the complex $\otimes_{j=1}^k (V_\bullet^j)$. The sheaves $\mathcal{T}or_i^Y(E_1, \dots, E_k)$ are independent of the choice of resolutions, hence they glue to a global twisted sheaf. If $\theta_j = \theta$ and $E_j = E$, for all j , denote $\mathcal{T}or_i^Y(E_1, \dots, E_k)$ by $\mathcal{T}or_i^Y(E, k)$ for short.

Let us check that $\mathcal{T}or_i^Y(E, k)$ is a θ^k -twisted sheaf. Fix coherent sheaves $E_1, \dots, E_{j-1}, E_{j+1}, \dots, E_k$ over $U_{\alpha\beta\gamma}$ and consider the endo-functor

$$\mathcal{T}or_i^Y(E_1, \dots, E_{j-1}, \bullet, E_{j+1}, \dots, E_k) : \text{Coh}(U_{\alpha\beta\gamma}) \longrightarrow \text{Coh}(U_{\alpha\beta\gamma})$$

of the category of coherent sheaves. Given a homomorphism $f : F \rightarrow G$ of coherent sheaves and a holomorphic function λ we get the equality

$$(2.6) \quad \mathcal{T}or_i^Y(E_1, \dots, E_{j-1}, \lambda f, E_{j+1}, \dots, E_k) = \lambda \mathcal{T}or_i^Y(E_1, \dots, E_{j-1}, f, E_{j+1}, \dots, E_k)$$

of homomorphisms from $\mathcal{T}or_i^Y(E_1, \dots, E_{j-1}, F, E_{j+1}, \dots, E_k)$ to $\mathcal{T}or_i^Y(E_1, \dots, E_{j-1}, G, E_{j+1}, \dots, E_k)$. We get the following equality of endomorphisms of the sheaf $\mathcal{T}or_i^Y(E_{\alpha|U_{\alpha\beta\gamma}}, k)$.

$$\mathcal{T}or_i^Y(g_{\alpha\beta}, k) \circ \mathcal{T}or_i^Y(g_{\beta\gamma}, k) \circ \mathcal{T}or_i^Y(g_{\gamma\alpha}, k) = \mathcal{T}or_i^Y(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}, k) = \mathcal{T}or_i^Y(\theta_{\alpha\beta\gamma} \cdot id, k) = \theta_{\alpha\beta\gamma}^k \cdot id,$$

where the last equality follows from equation (2.6).

Let $\theta \in Z^2(\mathcal{U}, \mathcal{O}_Y^*)$ be a two cocycle and $E := (E_\alpha, g_{\alpha\beta})$ a θ -twisted torsion free sheaf of rank $r > 0$. The sheaves

$$\mathcal{T}or_i^Y(E, r) \otimes \det(E)^{-1}$$

are θ^0 -twisted, so untwisted coherent sheaves. Let

$$\text{Sqrt}_r(x) := r + \frac{1}{r^r}(x - r^r) + \dots$$

be the Taylor series of the branch of the r -th root function centered at r^r . Set

$$(2.7) \quad \kappa(E) := \text{Sqrt}_r \left(\sum_{i=0}^{\dim(Y)} (-1)^i \text{ch} [\mathcal{T}or_i^Y(E, r) \otimes \det(E)^{-1}] \right).$$

If E is untwisted, then the class $\sum_{i=0}^{\dim(Y)} (-1)^i [\mathcal{T}or_i^Y(E, r)]$ in the K -group of Y represents the same class as that of the r -th power of the class of E . In that case the class given in equation (2.7) is equal to $\text{ch}(E) \exp(-c_1(E)/r)$, since $\text{ch} : K_{\text{top}} Y \rightarrow H^*(Y, \mathbb{Q})$ is a ring homomorphism.

Note: The Chern character $\text{ch}(F)$ of a θ -twisted sheaf, with a topologically trivial class θ , was defined in [HSt], depending on a choice of a lift of θ to a class in $H^2(Y, \mathbb{Q})$. Another definition is provided in [Li].

2.3. Sheaves of Azumaya algebras and their characteristic classes.

Definition 2.3. A *reflexive sheaf of Azumaya*⁴ \mathcal{O}_X -algebras of rank r over a Kähler manifold X is a sheaf E of reflexive coherent \mathcal{O}_X -modules, with a global section 1_E , and an associative multiplication $m : E \otimes E \rightarrow E$ with identity 1_E , admitting an open covering $\{U_\alpha\}$ of X , and an isomorphism $\eta_\alpha : E|_{U_\alpha} \rightarrow \mathcal{E}nd(F_\alpha)$ of unital associative algebras, for some reflexive sheaf F_α of rank r , over each U_α .

From now on the term a sheaf of Azumaya algebras will mean a reflexive sheaf of Azumaya \mathcal{O}_X -algebras. Fix a closed analytic subset $Z \subset X$, of codimension ≥ 3 , and set $U := X \setminus Z$. A reflexive sheaf of Azumaya \mathcal{O}_X -algebras is determined by its restriction to U . Hence, the set of isomorphism classes of reflexive Azumaya \mathcal{O}_X -algebras E of rank r , which are locally free over U , is in natural bijection with $H_{an}^1(U, PGL(r))$ [Mi]. Similarly, $H_{an}^1(U, PGL(r))$ parametrizes equivalence classes of coherent reflexive twisted \mathcal{O}_X -modules, which are locally free over U . We get a natural identification, of the set of isomorphism classes of reflexive sheaves of Azumaya \mathcal{O}_X -algebras, with the set of equivalence classes of coherent reflexive twisted \mathcal{O}_X -modules.

Let E be a reflexive sheaf of Azumaya \mathcal{O}_X -algebras, m its multiplication, and F a reflexive coherent twisted sheaf representing the equivalence class of (E, m) . We set

$$\kappa(E, m) := \kappa(F).$$

Caution 2.4. Note that $\kappa(E, m)$ is not equal to the class $\kappa(E)$ of the rank r^2 coherent sheaf E .

3. THE RATIONAL HODGE CLASSES $\kappa_i(X)$

Let S be a projective K3 surface and $v \in K_{top}S$ a primitive class of rank > 0 with $c_1(v)$ of type $(1, 1)$. Assume that $(v, v) \geq 2$. There is a system of hyperplanes in the ample cone of S , called v -walls, that is countable but locally finite [HL], Ch. 4C. An ample class is called v -generic, if it does not belong to any v -wall. Choose a v -generic ample class H . Then the moduli space $\mathcal{M}_H(v)$ is a projective irreducible holomorphic symplectic manifold, deformation equivalent to $S^{[n]}$, with $n = 1 + \frac{(v, v)}{2}$. This result is due to several people, including Huybrechts, Mukai, O'Grady, and Yoshioka. It can be found in its final form in [Y1].

Let f_1 and f_2 be the projections on the first and second factors of $S \times \mathcal{M}_H(v)$. Assume further that a universal sheaf \mathcal{E} exists over $S \times \mathcal{M}_H(v)$. Let $e : K_{top}S \rightarrow K_{top}\mathcal{M}_H(v)$ be the homomorphism given by

$$(3.1) \quad e_x := f_{2!} \left(f_1^!(-x^\vee) \otimes [\mathcal{E}] \right).$$

The class e_x has rank (v, x) , in terms of the Mukai pairing

$$(3.2) \quad (x, y) := -\chi(x^\vee \otimes y),$$

for $x, y \in K_{top}S$. Let v^\perp be the sublattice of $K_{top}S$ orthogonal to v .

Mukai defines a weight 2 Hodge structure on $K_{top}S \otimes_{\mathbb{Z}} \mathbb{C}$ as follows. The $(2, 0)$ summand is the pull-back of $H^{2,0}(S)$, via the Chern character isomorphism $ch : K_{top}S \rightarrow H^*(S, \mathbb{Z})$, and the

⁴Caution: The standard definition of a sheaf of Azumaya \mathcal{O}_X -algebras assumes that E is a locally free \mathcal{O}_X -module, while we assume only that it is reflexive.

pullback of $H^0(S, \mathbb{Z})$ and $H^4(S, \mathbb{Z})$ are both of Hodge-type $(1, 1)$. Recall that $H^2(\mathcal{M}_H(v), \mathbb{Z})$ is endowed with the Beauville-Bogomolov pairing. The homomorphism

$$(3.3) \quad \begin{aligned} v^\perp &\rightarrow H^2(\mathcal{M}_H(v), \mathbb{Z}), \\ x &\mapsto c_1(e_x), \end{aligned}$$

is an isometry and an isomorphism of weight 2 Hodge structures [Y1].

The *Mukai vector* of a class $v \in K_{top}S$ is the class $ch(v)\sqrt{td_S} \in H^*(S, \mathbb{Z})$. Following Mukai, we write the Mukai vector of v as a triple $(r, c_1(v), s)$, where the rank r corresponds to the summand in $H^0(S, \mathbb{Z})$, while the summand in $H^4(S, \mathbb{Z})$ corresponds to the integer s times the class Poincare-dual to a point. The Hirzebruch-Riemann-Roch Theorem yields the equality

$$(v, v) = c_1(v)^2 - 2rs.$$

Proposition 3.1. *The class $\kappa(e_v)$ is invariant under a finite-index subgroup⁵ of $Mon(\mathcal{M}_H(v))$. Let i be an integer satisfying $4 \leq 2i \leq n+2$. If i is even, then $\kappa_i(e_v)$ is $Mon(\mathcal{M}_H(v))$ -invariant. If i is odd, then the line $\text{span}_{\mathbb{Q}}\{\kappa_i(e_v)\}$ in $H^{2i}(\mathcal{M}_H(v), \mathbb{Q})$ is $Mon(\mathcal{M}_H(v))$ -invariant.*

The proposition is proven in section 5 using results of [Ma2, Ma4]. Proposition 3.1 yields a monodromy-invariant pair of a class and its negative, denoted by

$$(3.4) \quad \pm \kappa_i(X),$$

for any irreducible holomorphic symplectic manifold X of $K3^{[n]}$ -type, $n \geq 2$, and for $4 \leq 2i \leq n+2$. The class $\kappa_i(X)$ is of type (i, i) , by Lemma 3.2. Let X^d be the d -th cartesian product of X .

Lemma 3.2. *Let $\alpha \in H^{2i}(X^d, \mathbb{C})$ be a class, which is invariant under the diagonal action of a finite-index subgroup of $Mon(X)$. Then α is of Hodge type (i, i) .*

Proof. The case $d = 1$ of the statement is proven in [Ma4], Proposition 3.8 part 3. We sketch the proof for the convenience of the reader. We endow $Mon(X)$ with the Zariski topology induced by $GL(H^*(X, \mathbb{C}))$. The Lie algebra \mathfrak{g} of the identity component of the Zariski closure of $Mon(X)$ in $GL[H^*(X, \mathbb{C})]$ is equal to a faithful representation of $\mathfrak{so}(23)$ on $H^*(X, \mathbb{C})$, constructed by Verbitsky [Ve1, LL]. The equality of these Lie algebras is proven in [Ma2], Lemma 4.11. Verbitsky proved that the semi-simple endomorphism h of $H^*(X, \mathbb{C})$, which acts on $H^{p,q}(X)$ by $\sqrt{-1}(p - q)$, is an element of the image of $\mathfrak{so}(23)$, and is hence tangent to the identity component of the Zariski closure of $Mon(X)$. The latter component is also the identity component of the Zariski closure of any finite-index subgroup of $Mon(X)$, and in particular of the subgroup leaving the class α invariant. Hence, α belongs to the kernel of $\delta(h)$, where $\delta : \mathfrak{g} \rightarrow \mathfrak{gl}[H^*(X^d, \mathbb{C})]$ is the diagonal representation. Now $\delta(h) = \sum_{i=1}^d id_{X^{i-1}} \otimes h \otimes id_{X^{d-i}}$, which is the Hodge operator of $H^*(X^d, \mathbb{C})$. Hence, α is of Hodge type (i, i) . \square

Recall that the first main goal of this note is to express the Hodge-classes $\kappa_i(X)$ as characteristic classes of sheaves on X (Theorem 1.3). The generic such X does not contain any proper closed positive-dimensional analytic subvariety [Ve2].

⁵ Conjecturally, any monodromy operator takes $\kappa(e_v)$ to itself or its dual $\kappa((e_v)^\vee)$. The conjecture is proven in [Ma4] Corollary 1.6, if n is congruent to 0 or 1 modulo 4

3.1. Relation of Theorem 1.3 with the Hodge conjecture. Let X be a compact Kähler manifold. Denote by $Hodge(X)$ the \mathbb{Q} -subalgebra of $H^*(X, \mathbb{Q})$, generated rational (p, p) -classes, and let $Chern(X)$ be the \mathbb{Q} -subalgebra of $Hodge(X)$ generated by Chern classes of coherent sheaves on X . The classes $\kappa_i(E)$, of a θ -twisted torsion free sheaf on X , all belong to $Chern(X)$, by equation (2.7).

When X is a projective variety, the Hodge Conjecture predicts the equality $Chern(X) = Hodge(X)$. The extension of the Hodge Conjecture to Kähler manifolds *fails*; Voisin proved that there exist four-dimensional complex tori X , for which $Chern(X) \neq Hodge(X)$ [Vo].

4. MONODROMY INVARIANT CLASSES OVER $X \times X$

In section 4.1 we construct a reflexive sheaf over $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$, singular along the diagonal, which is, roughly, the canonical representative of a relative version of the class e_v . We resolve this sheaf as a locally free sheaf V , over the blow-up of the diagonal in $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$, in sections 4.1 and 4.2.

4.1. Monodromy invariant classes $\kappa_i(\mathcal{F})$ over $\mathcal{M}(v) \times \mathcal{M}(v)$. Set $\mathcal{M} := \mathcal{M}_H(v)$. Assume that a universal sheaf \mathcal{E} exists over $S \times \mathcal{M}$. (This assumption will be dropped later). A choice of a stable sheaf E in \mathcal{M} yields a lift of the class e_v , given in (3.1), to a class in the bounded derived category of coherent sheaves $D_{Coh}^b(\mathcal{M})$. Avoiding such a choice, we construct instead a natural class over $\mathcal{M} \times \mathcal{M}$.

Let π_{ij} be the projection from $\mathcal{M} \times S \times \mathcal{M}$ onto the product of the i -th and j -th factors. Consider the following object in the bounded derived category of coherent sheaves over $\mathcal{M} \times \mathcal{M}$:

$$(4.1) \quad \mathcal{F} := R_{\pi_{13}*} \left[\pi_{12}^* \mathcal{E}^\vee \otimes^L \pi_{23}^* \mathcal{E} \right] [1].$$

Let $\iota_E : \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}$, be the embedding sending a point $[E'] \in \mathcal{M}$ to (E, E') . Then ι_E relates the class of \mathcal{F} in $K_{top}(\mathcal{M} \times \mathcal{M})$ to e_v :

Lemma 4.1. $e_v = \iota_E^! [\mathcal{F}].$

Proof. Denote by $\tilde{\iota}_E : S \times \mathcal{M} \hookrightarrow \mathcal{M} \times S \times \mathcal{M}$ the morphism given by $(x, E') \mapsto (E, x, E')$. The Cohomology and Base Change Theorem yields the second equality below:

$$\iota_E^! [-\mathcal{F}] = \iota_E^! \pi_{13!} (\pi_{12}^! \mathcal{E}^\vee \otimes \pi_{23}^! \mathcal{E}) = f_{2!} \tilde{\iota}_E^! (\pi_{12}^! \mathcal{E}^\vee \otimes \pi_{23}^! \mathcal{E}) = f_{2!} (f_1^! E^\vee \otimes \mathcal{E}) = -e_v. \quad \square$$

Proposition 4.2. *The class $\kappa(\mathcal{F})$ in $H^*(\mathcal{M} \times \mathcal{M}, \mathbb{Q})$ is invariant under the diagonal action of a finite-index subgroup of $Mon(\mathcal{M})$. Let i be an integer satisfying $4 \leq 2i \leq n+2$. If i is even, then $\kappa_i(\mathcal{F})$ is $Mon(\mathcal{M})$ -invariant. If i is odd, then the line $\text{span}_{\mathbb{Q}}\{\kappa_i(\mathcal{F})\}$ in $H^{2i}(\mathcal{M} \times \mathcal{M}, \mathbb{Q})$ is $Mon(\mathcal{M})$ -invariant.*

The proposition is proven in section 5 using results of [Ma2, Ma4]. Compare also with Theorem 4.4 below.

Lemma 4.3. $c_1(\mathcal{F}) = -\pi_1^* c_1(e_v) + \pi_2^* c_1(e_v).$

The lemma is proven in section 5. When v is the class of the ideal sheaf of a length n subscheme, and \mathcal{E} is the universal ideal sheaf, then $c_1(e_v)$ is half the class of the big diagonal in $S^{[n]}$ ([Ma4], Lemma 5.9).

The object \mathcal{F} fits in an exact triangle

$$\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E}) \rightarrow \mathcal{F} \rightarrow \mathcal{E}xt_{\pi_{13}}^2(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})[-1] \rightarrow \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})[1].$$

Furthermore, $\mathcal{E}xt_{\pi_{13}}^2(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ is isomorphic to the structure sheaf \mathcal{O}_Δ of the diagonal $\Delta \subset [\mathcal{M} \times \mathcal{M}]$, while $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ is a reflexive sheaf of rank (v, v) , and is locally free away from Δ (Proposition 4.5). The object \mathcal{F} plays a central role in the study of the cohomology of the moduli space \mathcal{M} . It was used by Mukai to prove that if \mathcal{M} is 2-dimensional then it is a K3 surface [Mu2]. In the higher dimensional case properties of \mathcal{F} lead to a simple proof of the irreducibility of \mathcal{M} [KLS], originally proven by a degeneration argument; via a combined effort of several authors ([O'G], [Y1] Theorem 8.1, and [Y2] Corollary 3.15).

Theorem 4.4. (1) ([Ma1], Theorem 1) *The Chern classes of \mathcal{F} satisfy: $c_{2n-1}(\mathcal{F}) = 0$, and $c_{2n}(\mathcal{F})$ is Poincare-Dual to the class $[\Delta]$ of the diagonal. Hence, $c_{2n}(\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E}))$ is Poincare-Dual to $(1 - (2n - 1)!) [\Delta]$.*
 (2) ([Ma3], Theorem 1) *Consequently, the Chern classes $c_i(e_x)$, of the Künneth factors $e_x \in K_{top}\mathcal{M}$, $x \in K_{top}S$, of \mathcal{E} , given in (3.1), generate the integral cohomology ring $H^*(\mathcal{M}, \mathbb{Z})$.*

Generators for the cohomology ring, with *rational* coefficients, were found in [LQW, Ma1]. The ring structure was determined in [LS].

Let $\beta : B \rightarrow [\mathcal{M} \times \mathcal{M}]$ be the blow-up of $\mathcal{M} \times \mathcal{M}$ along Δ , $D := \mathbb{P}(T\Delta)$ the exceptional divisor, $\iota : D \hookrightarrow B$ the closed immersion, $\delta : \Delta \hookrightarrow \mathcal{M} \times \mathcal{M}$ the diagonal embedding, $p : D \rightarrow \Delta$ the bundle map, ℓ the tautological line-sub-bundle of $p^*T\Delta$, and ℓ^\perp the symplectic-orthogonal subbundle of $p^*T\Delta$. Let τ be the involution of $\mathcal{M} \times \mathcal{M}$, interchanging the two factors, and $\tilde{\tau}$ the induced involution of B . Note that $\tau^*(\mathcal{F}) = \mathcal{F}^\vee$, by Grothendieck-Serre's Duality, and the triviality of the relative canonical line-bundle $\omega_{\pi_{13}}$.

Proposition 4.5. (1) *The sheaf $E := \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ is reflexive of rank (v, v) .*
 (2) *E restricts to $[\mathcal{M} \times \mathcal{M}] \setminus \Delta$ as a locally free sheaf. We have the following isomorphism:*

$$(4.2) \quad \delta^*E \cong \left(\bigwedge^2 T^*\mathcal{M} \right) / \mathcal{O}_\mathcal{M} \cdot \sigma,$$

where σ is the symplectic form. For $i > 0$, we have

$$(4.3) \quad \mathcal{T}or_i^{\mathcal{M} \times \mathcal{M}}(E, \delta_*\mathcal{O}_\mathcal{M}) \cong \delta_* \bigwedge^{i+2} T^*\mathcal{M}.$$

(3) *The quotient*

$$(4.4) \quad V := [\beta^*\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})](D)/\text{tor},$$

by the torsion subsheaf, is a locally free sheaf of rank (v, v) over B .

(4) $\beta_*(V) \cong \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ and $R_{\beta_*}^i(V) = 0$, for $i > 0$.

(5) $\tilde{\tau}^*V$ is isomorphic to V^* .

(6) *The restriction $V|_D$ is naturally identified with the sub-quotient*

$$(4.5) \quad [\ell^\perp / \ell].$$

In particular, $V|_D$ is a symplectic vector bundle.

4.2. Proof of Proposition 4.5. The following lemma will be used in the proof of Proposition 4.5.

Lemma 4.6. *The following natural homomorphism is surjective:*

$$(4.6) \quad p^* p_* \left([\ell^\perp / \ell] \otimes \ell^* \right) \rightarrow [\ell^\perp / \ell] \otimes \ell^*.$$

Proof. We identify each of the vector bundles $T\Delta$ and $[\ell^\perp / \ell]$ with its dual, via the symplectic forms. We have the short exact sequence

$$0 \rightarrow [\ell^\perp / \ell] \otimes \ell^* \rightarrow [p^* T^* \Delta / \ell] \otimes \ell^* \rightarrow \ell^{-2} \rightarrow 0.$$

$p_* ([\ell^\perp / \ell] \otimes \ell^*) \cong \ker [p_* (\{p^* T^* \Delta \otimes \ell^*\} / \mathcal{O}) \rightarrow p_* (\ell^{-2})]$, which is naturally isomorphic to the quotient $[\wedge^2 T^* \Delta] / \mathcal{O}$, by the line-sub-bundle spanned by the symplectic form. The homomorphism (4.6) is dual to the wedge product $[\ell^\perp / \ell] \otimes \ell \rightarrow p^* ([\wedge^2 T^* \Delta] / \mathcal{O})$, which is clearly injective. \square

The proof of Proposition 4.5 requires a review of the following construction carried out in [Ma1]. There exists a (non-canonical) complex

$$(4.7) \quad V_{-1} \xrightarrow{g} V_0 \xrightarrow{f} V_1,$$

of locally free sheaves over $\mathcal{M} \times \mathcal{M}$, representing the object \mathcal{F} [Lan]. The sheaf homomorphism g is injective, since $\mathcal{E}xt_{\pi_{13}}^0(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})$ vanishes. The middle cohomology sheaf $\ker(f) / \text{Im}(g)$ is isomorphic to $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})$, and $\text{coker}(f)$ is isomorphic to $\mathcal{E}xt_{\pi_{13}}^2(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})$, and hence also to $\delta_* \mathcal{O}_\Delta$. Furthermore, the dual complex represents the pullback $\tau^*(\mathcal{F})$ of the object \mathcal{F} . In particular, $\text{coker}(g^*)$ is also isomorphic to $\delta_* \mathcal{O}_\Delta$.

Claim 4.7.

$$(4.8) \quad \mathcal{E}xt^1(\text{Im}(f), \mathcal{O}_{\mathcal{M} \times \mathcal{M}}) = 0,$$

$$(4.9) \quad \ker(f)^* \cong \text{coker}(f^*),$$

$$(4.10) \quad \ker(g^*)^* \cong \text{coker}(g).$$

Proof. Consider the long exact sequence of extension sheaves, obtained by applying $\mathcal{H}om(\bullet, \mathcal{O}_{\mathcal{M} \times \mathcal{M}})$ to the short exact sequence

$$0 \rightarrow \text{Im}(f) \rightarrow V_1 \rightarrow \mathcal{O}_\Delta \rightarrow 0.$$

$\mathcal{E}xt^i(V_1, \mathcal{O}_{\mathcal{M} \times \mathcal{M}}) = 0$, for $i > 0$, and $\mathcal{E}xt^i(\mathcal{O}_\Delta, \mathcal{O}_{\mathcal{M} \times \mathcal{M}}) = 0$, for $0 \leq i < \dim(\mathcal{M}) = 2n$, by the Local Duality Theorem. The vanishing (4.8) follows.

Applying $\mathcal{H}om(\bullet, \mathcal{O}_{\mathcal{M} \times \mathcal{M}})$ to the short exact sequence

$$0 \rightarrow \ker(f) \rightarrow V_0 \rightarrow \text{Im}(f) \rightarrow 0,$$

we get the short exact sequence

$$0 \rightarrow V_1^* \xrightarrow{f^*} V_0^* \rightarrow \ker(f)^* \rightarrow 0,$$

by the vanishing (4.8). Equation (4.9) follows.

Equation (4.10) is the analogue of Equation (4.9) for the dual of the complex (4.7). \square

Let K be the kernel of $g|_{\Delta} : (V_{-1})|_{\Delta} \rightarrow (V_0)|_{\Delta}$ and F the image of $f|_{\Delta} : (V_0)|_{\Delta} \rightarrow (V_1)|_{\Delta}$. Then K and $(V_1)|_{\Delta}/F$ are both isomorphic to \mathcal{O}_{Δ} . Let U_{-1} be the subsheaf of $(\beta^*V_{-1})(D)$, whose sections restrict to D as sections of $[\iota_*(p^*K)](D)$. We get the short exact sequence:

$$(4.11) \quad 0 \rightarrow \beta^*V_{-1} \rightarrow U_{-1} \rightarrow [\iota_*(p^*K)](D) \rightarrow 0.$$

Define $U_1 \subset \beta^*V_1$ as the subsheaf, whose sections restrict to D as sections of $\iota_*(p^*F)$. It fits in the short exact sequence:

$$(4.12) \quad 0 \rightarrow U_1 \rightarrow \beta^*V_1 \rightarrow \iota_*(p^*\text{coker}(f)) \rightarrow 0.$$

We get the complex of vector bundles over B

$$U_{-1} \xrightarrow{\tilde{g}} \beta^*V_0 \xrightarrow{\tilde{f}} U_1,$$

where both \tilde{f} and \tilde{g}^* are surjective. Both U_{-1} and U_1 are locally free \mathcal{O}_B -modules. Set

$$(4.13) \quad V := \ker(\tilde{f})/\text{Im}(\tilde{g})$$

(we no longer regard (4.4) as a definition, but rather as an equality, to be proven below). Then V is locally free as well.

Claim 4.8. (1) $\beta_*(U_{-1}) \cong V_{-1}$, and $R_{\beta_*}^i(U_{-1}) = 0$, for $i > 0$.

(2) $\beta_*(U_1) \cong \text{Im}(f)$, and $R_{\beta_*}^i(U_1) = 0$, for $i > 0$.

(3) $\beta_*(\ker(\tilde{f})) \cong \ker(f)$, and $R_{\beta_*}^i(\ker(\tilde{f})) = 0$, for $i > 0$.

Proof. 1) The higher direct images $R_{p_*}^i(\mathcal{O}_D(D))$ vanish, for $i \geq 0$. This vanishing implies Part 1, using the long exact sequence of higher direct images via β , associated to the short exact sequence (4.11).

2) The push-forward $p_*\mathcal{O}_D$ is isomorphic to \mathcal{O}_{Δ} , and all the higher direct images vanish. Part 2 follows from the long exact sequence of higher direct images via β , associated to the short exact sequence (4.12).

Part 3 follows part 2 using the long exact sequence of higher direct images via β , associated to the short exact sequence

$$0 \rightarrow \ker(\tilde{f}) \rightarrow \beta^*V_0 \xrightarrow{\tilde{f}} U_1 \rightarrow 0.$$

□

Proof of Proposition 4.5:

Part 1) Applying $\mathcal{H}om(\bullet, \mathcal{O}_{\mathcal{M} \times \mathcal{M}})$ to the short exact sequence

$$0 \rightarrow \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E}) \rightarrow \text{coker}(g) \rightarrow \text{Im}(f) \rightarrow 0,$$

we get the short exact sequence

$$0 \rightarrow V_1^* \xrightarrow{f^*} \ker(g^*) \rightarrow [\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})]^* \rightarrow 0,$$

by the vanishing (4.8) and equation (4.10). Hence, $[\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})]^*$ is the middle sheaf cohomology of the complex dual to (4.7). The dual complex represents the object $\tau^*\mathcal{F}$, in the derived category, so the middle sheaf cohomology is the pullback $\tau^*\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$. Reflexivity now follows, by applying the above argument to the dual complex, since $\tau^2 = id$.

Part 4 follows from Claim 4.8 and the long exact sequence of higher direct images via β , associated to the short exact sequence

$$0 \rightarrow U_{-1} \xrightarrow{\tilde{g}} \ker(\tilde{f}) \rightarrow V \rightarrow 0.$$

Part 6) Let Z be the total space of the vector bundle $\mathrm{Hom}(V_{-1}, V_0)$, $h : Z \rightarrow X \times X$ the projection, $g' : h^*V_{-1} \rightarrow h^*V_0$ the tautological homomorphism, $Z_1 \subset Z$ the determinantal stratum, where the rank of g' is $\mathrm{rank}(V_{-1}) - 1$, and $g : X \times X \rightarrow Z$ the section given in (4.7). Z_1 is a smooth locally closed subvariety, whose normal bundle N_{Z_1} is isomorphic to $\mathrm{Hom}(\ker(g'|_{Z_1}), \mathrm{coker}(g'|_{Z_1}))$ [ACGH]. The diagonal Δ is the scheme theoretic inverse image $g^{-1}(Z_1)$. Hence, the homomorphism

$$dg : N_\Delta \longrightarrow g^*N_{Z_1} = \mathrm{Hom}(\ker(g|_\Delta), \mathrm{coker}(g|_\Delta))$$

is injective at every fiber of N_Δ . Δ is also the degeneracy locus of the homomorphism f given in (4.7), and $f \circ g = 0$. Thus, the image of dg is contained in $\mathrm{Hom}(\ker(g|_\Delta), \ker(f|_\Delta)/\mathrm{Im}(g|_\Delta))$. Now, $\ker(g|_\Delta) \cong \mathcal{O}_\Delta$ and $\ker(f|_\Delta)/\mathrm{Im}(g|_\Delta)$ is isomorphic to $T\Delta$, by the well known identification of $T\mathcal{M}$ with the relative extension sheaf $\mathcal{E}xt_{f_2}^1(\mathcal{E}, \mathcal{E})$. We conclude that dg factors through a homomorphism

$$dg : N_\Delta \longrightarrow T\Delta,$$

which is fiber-wise injective, and hence an isomorphism.

Over B we have the tautological line-sub-bundle $\eta : \mathcal{O}_D(D) \hookrightarrow p^*N_\Delta$ and the homomorphism $d(\beta^*g)$ is the composition $p^*(dg) \circ \eta$. It follows that the image of $d(\beta^*g)$ is $\ell \subset T\Delta$, by the definition of ℓ . On the other hand, the image of $d(\beta^*g)$ is precisely

$$\mathrm{Hom}(p^*\ker(g|_\Delta), \mathrm{Im}(\tilde{g}|_D)/\mathrm{Im}(\beta^*g|_D)).$$

These two descriptions of the image of $d(\beta^*g)$ provide a canonical isomorphism $\ell \cong \mathrm{Im}(\tilde{g}|_D)/\mathrm{Im}(\beta^*g|_D)$. We see that $V|_D$ is a sub-bundle of $[p^*T\Delta]/\ell$.

Repeating the above argument, for the dual of the complex (4.7) and for the homomorphism f^* , we get that $(V|_D)^*$ is a subspace of $[p^*T\Delta]/\ell$ as well (under the identification $T\Delta \cong T^*\Delta$, via the symplectic structure). Hence, $V|_D$ is isomorphic to ℓ^\perp/ℓ .

Part 3) The direct image $p_*[(V|_D)]$ vanishes, by part 6 and the vanishing of $p_*[\ell^\perp/\ell]$. Hence, $\beta_*[V(-D)]$ is isomorphic to β_*V . The natural homomorphism $\beta^*\beta_*[V(-D)] \rightarrow V(-D)$ is surjective, by part 6 and Lemma 4.6. Part 3 follows from part 4, since the kernel of the homomorphism $\beta^*\beta_*[V(-D)] \rightarrow V(-D)$ is supported on D , and is hence the torsion subsheaf of $\beta^*\beta_*[V(-D)]$.

Part 5) If we repeat the construction of the vector bundle (4.13), using the dual of the complex (4.7), we obtain the vector bundle V^* , by a direct check. On the other hand, the dual complex represents $\tau^*\mathcal{F}$, and the proof of the equality of the sheaves (4.4) and (4.13) yields the isomorphism

$$V^* \cong \beta^*[\tau^*\{\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})\}](D)/\mathrm{tor}.$$

The statement now follows from the equality $\beta^*\tau^* = \tilde{\tau}^*\beta^*$.

Part 2) Consider the exact triangle

$$E \xrightarrow{a} [V_{-1} \rightarrow V_0 \rightarrow V_1] \xrightarrow{b} \mathcal{O}_\Delta[-1] \rightarrow E[1].$$

Restriction to Δ yields the long exact sequence

$$\begin{array}{ccccccc}
\mathcal{T}or_2^{\mathcal{M} \times \mathcal{M}}(E, \mathcal{O}_\Delta) & \xrightarrow{a_{-2}} & 0 & \xrightarrow{b_{-2}} & \mathcal{T}or_3^{\mathcal{M} \times \mathcal{M}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\delta_{-1}} & \\
\mathcal{T}or_1^{\mathcal{M} \times \mathcal{M}}(E, \mathcal{O}_\Delta) & \xrightarrow{a_{-1}} & \mathcal{O}_\Delta & \xrightarrow{b_{-1}} & \mathcal{T}or_2^{\mathcal{M} \times \mathcal{M}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\delta_0} & \\
E \otimes \mathcal{O}_\Delta & \xrightarrow{a_0} & T\Delta & \xrightarrow{b_0} & \mathcal{T}or_1^{\mathcal{M} \times \mathcal{M}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) & \xrightarrow{\delta_1} & \\
0 & \xrightarrow{a_1} & \mathcal{O}_\Delta & \xrightarrow{b_1} & \mathcal{O}_\Delta \otimes \mathcal{O}_\Delta & \rightarrow & 0.
\end{array}$$

Note that $\mathcal{T}or_i^{\mathcal{M} \times \mathcal{M}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ is isomorphic to $\bigwedge^i T^*\Delta$. Clearly, δ_{-i} is an isomorphism, for $i \geq 2$. The isomorphism in Equation (4.3) follows. The homomorphism b_0 is surjective, hence an isomorphism. Thus $a_0 = 0$ and δ_0 is surjective.

The isomorphism (4.2) would follow, once we prove that b_{-1} is injective. The proof is by contradiction. Assume that b_{-1} vanishes. Then δ_0 is injective and $\delta_0(\sigma)$ is a non-zero global section of $H^0(E \otimes \mathcal{O}_\Delta)$. Let $\text{tor}(\beta^*E)$ be the torsion subsheaf of β^*E . The endo-functor $R_{\beta_*}L_{\beta^*}$ of $D^b(\mathcal{M} \times \mathcal{M})$ is the identity. Hence, $\beta_*(\text{tor}(\beta^*E)) = 0$, since E is torsion free, by part 1. In particular, $H^0(\text{tor}(\beta^*E)) = 0$. Now $[\beta^*E/\text{tor}(\beta^*E)]|_D \cong \ell^\perp/\ell$, by part 6, and $H^0(\ell^\perp/\ell) = 0$. Thus, $H^0(D, [\beta^*E]|_D) = 0$. Consequently, $H^0(E \otimes \mathcal{O}_\Delta) = 0$. A contradiction. This completes the proof of Proposition 4.5. \square

4.3. Lifting deformations of a moduli space \mathcal{M} to deformation of the pair $(\mathcal{M}, \mathcal{E})$.

Let S' be another projective $K3$ surface, $v' \in K_{\text{top}}S'$ a primitive class satisfying $(v', v') = 2n - 2$, $n \geq 2$, and H' a v' -generic ample line bundle H' . Assume that $\mathcal{M}' := \mathcal{M}_{H'}(v')$ is non-empty. Yoshioka proved that the moduli space \mathcal{M}' is an irreducible holomorphic symplectic variety, deformation equivalent to $S^{[n]}$ [Y1]. His proof implies the existence of a sequence of families of $K3$ -surfaces $\mathcal{S}_i \rightarrow T_i$, $1 \leq i \leq N$, over quasi-projective curves T_i , with smooth and proper relative families of such moduli spaces $\mathcal{M}_{\mathcal{S}_i/T_i}$ having the following properties. There exist points $t'_i \in T_i$ and $t''_{i+1} \in T_{i+1}$, and an isomorphism ϕ_i from the fiber $\mathcal{M}_{t'_i}$ onto the fiber $\mathcal{M}_{t''_{i+1}}$. Finally, $\mathcal{M}_{t'_1} = \mathcal{M}_{H'}(v')$, and $\mathcal{M}_{t''_N} = S^{[n]}$.

The isomorphism ϕ_i comes in two flavors. One is induced by a Fourier-Mukai transformations between the derived categories of $S_{t'_i}$ and $S_{t''_{i+1}}$ mapping stable sheaves to stable sheaves. Such Fourier-Mukai transformations relate a twisted universal sheaf over $S_{t'_i} \times \mathcal{M}_{t'_i}$ to one over $S_{t''_i} \times \mathcal{M}_{t''_i}$ ([Mu3], Theorem 1.6).

The second flavor is induced by the composition, of a Fourier-Mukai transformation, with the functor, which takes an object or a morphism, in the derived category, to its dual. The composite functor relates a twisted universal sheaf over $S_{t'_i} \times \mathcal{M}_{t'_i}$ to the *dual* of one over $S_{t''_{i+1}} \times \mathcal{M}_{t''_{i+1}}$.

The following Lemma thus follows from Yoshioka's work. It will play a central role in section 6. Let \mathcal{E} be the twisted sheaf over $\mathcal{M} \times \mathcal{M}$ in Proposition 4.5 and \mathcal{E}' its analogue over $\mathcal{M}' \times \mathcal{M}'$. Note that $\mathcal{E}nd(\mathcal{E})$ and $\mathcal{E}nd(\mathcal{E}^*)$ are isomorphic reflexive coherent sheaves, but they are not isomorphic as reflexive sheaves of Azumaya algebras.

Lemma 4.9. *The pair $(\mathcal{M}', \{\mathcal{E}nd(\mathcal{E}'), \mathcal{E}nd((\mathcal{E}')^*)\})$ deforms to the pair $(\mathcal{M}, \{\mathcal{E}nd(\mathcal{E}), \mathcal{E}nd(\mathcal{E}^*)\})$. The structures of Azumaya algebras deform as well.*

5. PROOF OF THE MONODROMY INVARIANCE OF $\kappa_i(X)$ AND $\kappa_i(\mathcal{F})$

We prove Propositions 3.1 and 4.2 and Lemma 4.3, after reviewing the necessary facts about the monodromy group of $S^{[n]}$.

Let S be a K3 surface, $v \in K_{top}S$ a primitive class with $c_1(v)$ of type $(1, 1)$, and H a v -generic line bundle. Assume that $\mathcal{M}_H(v)$ is non-empty (in particular, $\text{rank}(v) \geq 0$, $(v, v) \geq -2$, etc ...). Then $\mathcal{M}_H(v)$ is a projective irreducible holomorphic symplectic manifold of K3^[n]-type, with $2n = (v, v) + 2$. Assume that $(v, v) \geq 2$. Then $H^2(\mathcal{M}_H(v), \mathbb{Z})$, endowed with the Beauville-Bogomolov pairing, is Hodge isometric to $v^\perp \subset K_{top}S$, via Mukai's isometry (3.3).

We define next the orientation character of $O(K_{top}S)$. A 4-dimensional subspace V of $K_{top}S \otimes_{\mathbb{Z}} \mathbb{R}$ is *positive definite*, if the Mukai pairing restricts to V as a positive-definite pairing. The positive cone $\mathcal{C}_+ \subset K_{top}S \otimes_{\mathbb{Z}} \mathbb{R}$, given by

$$\mathcal{C}_+ := \{x : (x, x) > 0\},$$

is homotopic to the unit 3-sphere in any 4-dimensional *positive definite* subspace ([Ma6], Lemma 4.1). Hence $H^3(\mathcal{C}_+, \mathbb{Z})$ is isomorphic to \mathbb{Z} and is a natural character

$$cov : O(K_{top}S) \longrightarrow \{\pm 1\}$$

of the isometry group. Let $O^+(K_{top}S)$ be the kernel of cov .

Denote by $O(K_{top}S)_v$ the subgroup of isometries of $K_{top}S$, stabilizing v . Let g be any isometry in $O(K_{top}S)_v$. It is *not* assumed to preserve the Hodge structure. Denote by

$$(g \otimes 1) : K_{top}S \otimes K_{top}\mathcal{M}(v) \longrightarrow K_{top}S \otimes K_{top}\mathcal{M}(v)$$

the homomorphism acting via the identity on the second factor. The Künneth Theorem identifies $K_{top}S \otimes K_{top}\mathcal{M}(v)$ with $K_{top}[S \times \mathcal{M}(v)]$ ([A], Corollary 2.7.15). Assume that a universal sheaf \mathcal{E}_v exists over $S \times \mathcal{M}(v)$ and let $[\mathcal{E}_v]$ be its class in $K_{top}[S \times \mathcal{M}(v)]$. Let $D : K_{top}S \rightarrow K_{top}S$ be the involution, sending a class x to its dual x^\vee . Set $m := (v, v) + 2$. We define a class in the middle cohomology $H^{2m}(\mathcal{M}(v) \times \mathcal{M}(v), \mathbb{Z})$:

$$\overline{mon}(g) := \begin{cases} c_m(-\pi_{13!} \{ \pi_{12}^! [(g \otimes 1)[\mathcal{E}_{v_1}]]^\vee \cup \pi_{23}^! [\mathcal{E}_{v_2}] \}) & \text{if } cov(g) = 1, \\ c_m(-\pi_{13!} \{ \pi_{12}^! [(Dg \otimes 1)[\mathcal{E}_{v_1}]] \cup \pi_{23}^! [\mathcal{E}_{v_2}] \}) & \text{if } cov(g) = -1. \end{cases}$$

Denote by

$$(5.1) \quad mon(g) : H^*(\mathcal{M}(v), \mathbb{Z}) \longrightarrow H^*(\mathcal{M}(v), \mathbb{Z})$$

the homomorphism obtained from $\overline{mon}(g)$ using the Künneth and Poincare-Duality Theorems.

Theorem 5.1. ([Ma2] Theorems 1.2 and 1.6)

- (1) The endomorphism $mon(g)$ is an algebra automorphism and a monodromy operator.
- (2) The assignment

$$(5.2) \quad mon : O(K_{top}S)_v \longrightarrow Mon(\mathcal{M}(v)),$$

sending an isometry g to the operator $mon(g)$, is a group-homomorphism. The homomorphism is injective, if $(v, v) \geq 4$, and its kernel is generated by the involution

$$(5.3) \quad w \mapsto -w + (w, v)v,$$

if $(v, v) = 2$. The image $mon[O(K_{top}(S)_v)]$ is a normal subgroup of finite index in the monodromy group $Mon(\mathcal{M}_H(v))$.

- (3) *There exists a topological complex line bundle ℓ_g on $\mathcal{M}_H(v)$ satisfying one of the following equations:*

$$\begin{aligned} (g \otimes \text{mon}(g))[\mathcal{E}_v] &= [\mathcal{E}_v] \otimes f_2^* \ell_g, & \text{if } \text{cov}(g) = 1, \\ ((D \circ g) \otimes \text{mon}(g))[\mathcal{E}_v] &= [\mathcal{E}_v]^\vee \otimes f_2^* \ell_g, & \text{if } \text{cov}(g) = -1. \end{aligned}$$

The action of $\text{mon}(g)$ on $K_{\text{top}}(\mathcal{M}(v))$, in part 3 of the Theorem, is constructed as follows. The Chern character homomorphism $ch : K_{\text{top}}\mathcal{M}(v) \rightarrow H^*(\mathcal{M}(v), \mathbb{Q})$ is injective, since $K_{\text{top}}\mathcal{M}(v)$ is torsion free [Ma3]. The homomorphism ch is monodromy equivariant; hence it maps $K_{\text{top}}\mathcal{M}(v)$ to a $\text{mon}(g)$ -invariant subalgebra, for all $g \in O(K_{\text{top}}S)_v$, by part 1 of Theorem 5.1. Denote by mon_g the corresponding monodromy automorphism of $K_{\text{top}}\mathcal{M}(v)$. Part 3 of the Theorem can be rephrased in terms of the homomorphism $e : K_{\text{top}}S \rightarrow K_{\text{top}}\mathcal{M}(v)$, given in (3.1):

$$(5.4) \quad \text{mon}_g(e_{g^{-1}(x)}) = \begin{cases} e_x \otimes \ell_g, & \text{if } \text{cov}(g) = 1, \\ (e_x)^\vee \otimes \ell_g, & \text{if } \text{cov}(g) = -1. \end{cases}$$

Consequently, the line bundle ℓ_g is determined by the following formula:

$$(5.5) \quad c_1(\ell_g) = \frac{\text{mon}_g(c_1(e_v)) - \text{cov}(g) \cdot c_1(e_v)}{(v, v)}.$$

Let $\text{Mon}^2(\mathcal{M}(v))$ be the image in $O[H^2(\mathcal{M}(v), \mathbb{Z})]$ of $\text{Mon}(\mathcal{M}(v))$, under the restriction homomorphism from $H^*(\mathcal{M}(v), \mathbb{Z})$ to $H^2(\mathcal{M}(v), \mathbb{Z})$. Let

$$\text{mon}^2 : O(K_{\text{top}}S)_v \rightarrow \text{Mon}^2(\mathcal{M}(v))$$

be the composition of mon with the restriction homomorphism.

Theorem 5.2. ([Ma4], Theorem 1.2 and Lemma 4.2). *The homomorphism $\text{mon}^2 : O(K_{\text{top}}S)_v \rightarrow \text{Mon}^2(\mathcal{M}(v))$ is surjective. It is an isomorphism, if $(v, v) \geq 4$, and its kernel is generated by the involution (5.3), if $(v, v) = 2$.*

Theorem 5.3. ([Ma4] Theorem 1.5) *Any monodromy operator, which acts as the identity on $H^2(\mathcal{M}(v), \mathbb{Z})$, acts as the identity also on $H^k(\mathcal{M}(v), \mathbb{Z})$, $0 \leq k \leq \frac{\dim_{\mathbb{R}}(\mathcal{M}(v))}{4} + 2$.*

Proof of Proposition 3.1: Observe first that the pair $\{\kappa(e_v), \kappa((e_v)^\vee)\}$ is invariant under the image of $O(K_{\text{top}}S)_v$ in $\text{Mon}(\mathcal{M}(v))$ via mon . This follows from the reformulation (5.4) of part 3 of Theorem 5.1, and the fact that $g(v) = v$. We prove next the $\text{Mon}(\mathcal{M}(v))$ -invariance of $\text{span}_{\mathbb{Q}}\{\kappa_i(e_v)\}$, for $4 \leq 2i \leq n + 2$. Let f be an element of $\text{Mon}(\mathcal{M}(v))$. There exists an isometry $g \in O(K_{\text{top}}S)_v$, such that f and $\text{mon}(g)$ have the same image in $\text{Mon}^2(\mathcal{M}(v))$, by Theorem 5.2. Theorem 5.3 implies that the actions of f and $\text{mon}(g)$ on $H^k(\mathcal{M}(v), \mathbb{Z})$ agree in the range required. We conclude the f -invariance of $\text{span}_{\mathbb{Q}}\{\kappa_i(e_v)\}$, from its $\text{mon}(g)$ invariance, for i in that range. \square

Proof of Proposition 4.2: It suffices to show that the pair $\{\kappa(\mathcal{F}), \kappa(\mathcal{F}^\vee)\}$ is invariant under the image of $O(K_{\text{top}}S)_v$ in $\text{Mon}(\mathcal{M}(v))$ via mon , by the argument used in the proof of Proposition 3.1. Denote by

$$D_{\mathcal{M}} : K_{\text{top}}\mathcal{M}(v) \rightarrow K_{\text{top}}\mathcal{M}(v)$$

the duality involution $y \mapsto y^\vee$ and by D_S the duality involution of $K_{\text{top}}S$. Note that $[\mathcal{E}_v]^\vee = (D_S \otimes D_{\mathcal{M}})[\mathcal{E}_v]$. Caution: while $D_{\mathcal{M}}$ commutes with $\text{Mon}(\mathcal{M}(v))$, D_S does *not* commute with

$O(K_{top}S)_v$. The class $[\mathcal{F}]$ is the image in $K_{top}\mathcal{M}(v) \otimes K_{top}\mathcal{M}(v)$ of the class $\{(1 \otimes D_{\mathcal{M}})[\mathcal{E}_v]\} \otimes [\mathcal{E}_v]$ via the contraction with the Mukai pairing:

$$\begin{aligned} \psi : [K_{top}S \otimes K_{top}\mathcal{M}(v)] \otimes [K_{top}S \otimes K_{top}\mathcal{M}(v)] &\rightarrow K_{top}\mathcal{M}(v) \otimes K_{top}\mathcal{M}(v) \\ x_1 \otimes y_1 \otimes x_2 \otimes y_2 &\mapsto -\chi(x_1^\vee \otimes x_2)y_1 \otimes y_2 \end{aligned}$$

The equality $\psi = \psi \circ (g \otimes 1 \otimes g \otimes 1)$ holds, for any isometry g of the Mukai lattice. Hence, the following equality holds:

$$\psi \{(g \otimes mon_g \circ D_{\mathcal{M}})[\mathcal{E}_v] \otimes (g \otimes mon_g)[\mathcal{E}_v]\} = \psi \{(1 \otimes mon_g \circ D_{\mathcal{M}})[\mathcal{E}_v] \otimes (1 \otimes mon_g)[\mathcal{E}_v]\}.$$

The right hand side is $(mon_g \otimes mon_g)[\mathcal{F}]$, while the left hand side is equal to

$$\begin{cases} \psi \{(1 \otimes D_{\mathcal{M}})([\mathcal{E}_v] \otimes f_2^! \ell_g) \otimes [\mathcal{E}_v] \otimes f_2^! \ell_g\}, & \text{if } cov(g) = 1, \\ \psi \{([\mathcal{E}_v] \otimes f_2^! \ell_g^\vee) \otimes (1 \otimes D_{\mathcal{M}})([\mathcal{E}_v] \otimes f_2^! \ell_g^\vee)\}, & \text{if } cov(g) = -1, \end{cases}$$

by part 3 of Theorem 5.1. The latter contractions simplify to

$$(5.6) \quad (mon_g \otimes mon_g)[\mathcal{F}] = \begin{cases} [\mathcal{F}] \otimes \pi_1^!(\ell_g^\vee) \otimes \pi_2^! \ell_g, & \text{if } cov(g) = 1, \\ ([\mathcal{F}])^\vee \otimes \pi_1^!(\ell_g^\vee) \otimes \pi_2^! \ell_g, & \text{if } cov(g) = -1, \end{cases}$$

by the projection formula. We conclude that the pair $\{\kappa(\mathcal{F}), \kappa(\mathcal{F}^\vee)\}$ is $mon(g)$ -invariant. \square

Proof of Lemma 4.3: For every $g \in O^+(K_{top}S)_v$, we have:

$$\begin{aligned} (mon_g \otimes mon_g)(c_1(\mathcal{F})) &\stackrel{(5.6)}{=} c_1(\mathcal{F}) - (v, v)[\pi_1^* c_1(\ell_g) - \pi_2^* c_1(\ell_g)] \stackrel{(5.5)}{=} \\ &c_1(\mathcal{F}) - \pi_1^*[mon_g(c_1(e_v)) - c_1(e_v)] + \pi_2^*[mon_g(c_1(e_v)) - c_1(e_v)]. \end{aligned}$$

Consequently, $c_1(\mathcal{F}) + \pi_1^*(c_1(e_v)) - \pi_2^* c_1(e_v)$ is $O^+(K_{top}S)_v$ -invariant. The $O^+(K_{top}S)_v$ -invariant subspace of $H^2(\mathcal{M}(v) \times \mathcal{M}(v))$ vanishes, since the latter is the direct sum of two copies of the non-trivial irreducible $O^+(K_{top}S)_v$ -module $H^2(\mathcal{M}(v))$. \square

6. HYPERHOLOMORPHIC SHEAVES

We review Verbitsky's theory of hyperholomorphic reflexive sheaves [Ve4]. It plays a central role in the proof of Theorem 1.6.

6.1. Twistor deformations of pairs. Let X be an irreducible holomorphic-symplectic manifold, ω a Kähler class of X , and $\mathcal{X} \rightarrow \mathbb{P}_\omega^1$ the associated twistor deformation [HKLR, Hu]. Recall that associated to ω and the complex structure I is a Ricci-flat hermitian metric g , by the Calabi-Yau theorem [Be]. Furthermore, any two among I , ω , and g , determine the third. The twistor deformation $\mathcal{X} \rightarrow \mathbb{P}_\omega^1$ comes with a canonical differentiable trivialization $\mathcal{X} \cong X \times \mathbb{P}_\omega^1$. The hermitian metric on X is constant with respect to this trivialization, but the complex structure I_t and the associated Kähler form ω_t vary as we vary $t \in \mathbb{P}_\omega^1$. We denote by X_t the differentiable manifold X endowed with the complex structure I_t . We denote by $0 \in \mathbb{P}_\omega^1$ the point corresponding to the complex structure I on X .

Let F be a reflexive sheaf on X and $(F)_{sing}$ the singular locus of F . Then $(F)_{sing}$ has codimension ≥ 3 in X . Set $(F)_{sm} := X \setminus (F)_{sing}$. Let g_F be a hermitian metric on the restriction of F to $(F)_{sm}$. Associated to g_F and the holomorphic structure $\bar{\partial}$ of F is the Chern connection ∇ ([GH], Ch. 0 Sec. 5, Lemma page 73). Recall that $\bar{\partial}$ is the $(0, 1)$ -part of ∇ . The decomposition $T_{\mathbb{C}}^*X := T^{1,0}X \oplus T^{0,1}X$, of the complexified cotangent bundle of X , depends on the complex structure I of X .

When the sheaf F is ω -slope-stable, then there exists a unique Hermite-Einstein metric g_F , whose curvature form is L^2 -integrable, on the restriction of F to $(F)_{sm}$ [BS]. We will refer to g_F as the *Hermite-Einstein metric* of F and to its Chern connection as the *Hermite-Einstein connection* of F . Denote by $\bar{\partial}_t$, $t \in \mathbb{P}_\omega^1$, the $(0, 1)$ -part of ∇ with respect to the complex structure I_t . Then $\bar{\partial}_0^2 = 0$, but $\bar{\partial}_t^2$ need not vanish for a general $t \in \mathbb{P}_\omega^1$.

Definition 6.1. ([Ve4], Definition 3.15) An ω -slope-stable reflexive sheaf F over (X, ω) is ω -stable-hyperholomorphic, if $\bar{\partial}_t^2 = 0$, for all $t \in \mathbb{P}_\omega^1$. An ω -slope-polystable reflexive sheaf F is ω -polystable-hyperholomorphic, if each ω -slope-stable direct summand of F is ω -stable-hyperholomorphic.

Definition 6.2. ([Ve4], Definition 2.9) A subvariety Z of X is ω -tri-analytic, if the canonical differentiable trivialization $\mathcal{X} \cong X \times \mathbb{P}_\omega^1$ maps $Z \times \mathbb{P}_\omega^1$ to a closed analytic subvariety of \mathcal{X} .

Verbitsky proves that the singularity locus $(F)_{sing}$, of a reflexive hyperholomorphic sheaf, is supported over a tri-analytic⁶ subvariety of X ([Ve4], Claim 3.16). The complex structure $\bar{\partial}_t$ on F defines a locally free \mathcal{O}_{X_t} -module over $X_t \setminus (F)_{sing}$. We denote by F_t the reflexive sheaf on X_t corresponding to the push-forward of the latter via the inclusion into X_t . In particular, $F_0 = F$. The following is a fundamental result of Verbitsky.

Theorem 6.3. ([Ve4], Theorem 3.19) Let E be an ω -slope-stable reflexive sheaf on X . Assume that $c_i(E)$ is of Hodge-type (i, i) , for $i = 1, 2$, and for all complex structures parametrized by the twistor line \mathbb{P}_ω^1 . Then E is ω -stable hyperholomorphic.

The notion of ω -slope-stability is well defined for twisted sheaves as well. Slope-stability of a torsion-free sheaf E depends on the sheaf $\mathcal{E}nd(E)$ of Lie-algebras and its subsheaves of maximal parabolic subalgebras. Given a subsheaf F of E , the condition $\text{slope}_\omega(F) < \text{slope}_\omega(E)$ is equivalent to

$$(6.1) \quad \deg_\omega(\mathcal{H}om(E, F)) < 0.$$

The sheaf $\mathcal{H}om(E, F)$ is untwisted, for every θ -twisted subsheaf F of a θ -twisted sheaf E .

Definition 6.4. (1) Let E be a torsion free θ -twisted sheaf and ω a Kähler class on X . We say that E is ω -slope-stable, if the inequality (6.1) holds, for every non-zero θ -twisted proper subsheaf F of E . The sheaf E is ω -slope-semi-stable, if the analogue of (6.1), with strict inequality replaced by \leq , holds for every such F . The sheaf E is said to be ω -slope-polystable, if it is ω -slope-semi-stable and away from a locus of codimension two E is isomorphic to a direct sum of ω -slope-stable sheaves.

(2) A reflexive sheaf A of Azumaya algebras (Definition 2.3) is ω -slope-stable (resp. ω -slope-polystable), if some, hence any lift of A to a twisted reflexive sheaf has the corresponding property. Equivalently, A is ω -slope-stable, if every non-trivial subsheaf of maximal parabolic subalgebras of A has negative ω -slope.

⁶Note that Z is tri-analytic if and only if Z is analytic with respect to I_t , for all $t \in \mathbb{P}_\omega^1$. The ‘only if’ direction is clear. The ‘if’ direction follows from the following fact. Given points $t \in \mathcal{P}_\omega^1$ and $x \in X$, we get the direct sum decomposition $T_{(x,t)}\mathcal{X} = T_t\mathbb{P}_\omega^1 \oplus T_xX$, of the real tangent space, induced by the differentiable trivialization of \mathcal{X} . The relevant fact is that both summands are complex subspaces, even though the projection $\mathcal{X} \rightarrow X$ is not holomorphic ([HKLR], formula (3.71)).

Note that if E is reflexive and ω -slope-polystable, then E is a direct sum of ω -slope-stable sheaves ([HL], Corollary 1.6.11).

Lemma 6.5. ([Ve4], section 3.5) *Let F and G be two reflexive ω -polystable-hyperholomorphic sheaves of ω -slope 0. Then the following statements hold.*

- (1) *Any global section f of F is flat with respect to the Hermite-Einstein connection. In particular, f is a holomorphic section with respect to all complex structures $\bar{\partial}_t$, $t \in \mathbb{P}_\omega$.*
- (2) *There exists a canonical isomorphism of vector spaces $\text{Hom}(F_t, G_t) \rightarrow \text{Hom}(F_s, G_s)$, for all $s, t \in \mathbb{P}_\omega$.*
- (3) *If F_t is endowed with an associative multiplication $m_t : F_t \otimes F_t \rightarrow F_t$, or more specifically a structure of an Azumaya \mathcal{O}_{X_t} -algebra, or a Lie-algebra structure $[\cdot, \cdot]_t : F_t \otimes F_t \rightarrow F_t$, then F_s is naturally endowed with such a structure, for all $s \in \mathbb{P}_\omega$.*
- (4) *Any saturated subsheaf F' of F of ω -slope zero is reflexive and ω -polystable-hyperholomorphic.*
- (5) *Let $\varphi : F \rightarrow G$ be a homomorphism. Then $\ker(\varphi)$ and the saturation of $\text{Im}(\varphi)$ are ω -polystable-hyperholomorphic.*
- (6) *Let F'_t be a saturated subsheaf of F_t , of ω_t -slope 0, for some $t \in \mathbb{P}_\omega$. Then F'_t extends to an ω -polystable-hyperholomorphic subsheaf F'_s of F_s , for all $s \in \mathbb{P}_\omega$.*
- (7) *If F_t has a structure of an Azumaya algebra and the subsheaf F'_t in part 6 is a maximal parabolic subalgebra, then the subsheaf F'_s is a maximal parabolic subalgebra, for all $s \in \mathbb{P}_\omega$.*

Proof. Parts 1 and 2) See [Ve4], Theorem 3.27.

Part 3) The sheaf $\mathcal{H}om(\mathcal{H}om(F_t^*, F_t), F_t)$ is ω_t -polystable-hyperholomorphic and m (or $[\cdot, \cdot]$) is a global section of this sheaf, hence a flat section with respect to the induced Hermite-Einstein connection, hence a holomorphic section with respect to all induced complex structures, by part 1. The axioms of the corresponding algebraic structure are expressed as identities involving flat sections. Hence they hold for all $s \in \mathbb{P}_\omega$, since they hold at t .

Part 4) The ω -slope-stable summands of F are hyperholomorphic, and F' is necessarily isomorphic to a direct sum of such summands.

Part 5) The kernel and image of φ must have ω -slope zero.

Part 6) The sheaf F'_t is ω_t -slope-polystable-hyperholomorphic, by part 4. The sheaf $\text{Hom}(F'_t, F_t) \otimes_{\mathbb{C}} F_s^*$ is ω_s -slope-polystable-hyperholomorphic, it is canonically isomorphic to $\text{Hom}(F'_s, F_s) \otimes_{\mathbb{C}} F_s^*$, by part 2, and the evaluation homomorphism from the latter into F_s has a hyperholomorphic image, by part 5.

Part 7) Assume that F'_t is a saturated ω_t -slope 0 Lie subalgebra. Then its extension, in part 6, consists of Lie subalgebras F'_s , for all $s \in \mathbb{P}_\omega$, by part 3. Let F''_s be the subsheaf orthogonal to L'_s with respect to the trace bilinear pairing. Then L'_s is a sheaf of maximal parabolic subalgebra over $(F)_{sm}$, if and only if the following two conditions hold: a) F''_s is a subsheaf of F'_s , and b) the homomorphism $F''_s \otimes F''_s \rightarrow F_s$, given by $a \otimes b \mapsto ab$, vanishes. Now F''_s is the kernel of the homomorphism $F'_s \rightarrow (F'_s)^*$, induced by the trace pairing. Hence F''_s is saturated of ω_s -slope 0, and so ω_s -polystable-hyperholomorphic, by part 4.

Assume now that F'_t is a saturated ω_t -slope 0 maximal parabolic subalgebra of F_t . We get a flag $F'' \subset F' \subset F$ of ω -polystable-hyperholomorphic reflexive sheaves of ω -slope 0. Furthermore, each of the conditions a) and b) above is expressed in terms of the vanishing of a natural homomorphism between ω -polystable-hyperholomorphic sheaves of slope zero. Hence, if they both hold for t , then they both hold for all $s \in \mathbb{P}_\omega$, by part 1. \square

6.2. Projectively ω -stable-hyperholomorphic sheaves. The theory of ω -slope-stable θ -twisted reflexive sheaves is incomplete, as the Uhlenbeck-Yau theorem, about the existence of Hermite-Einstein metric, is not available⁷ in this generality. Nevertheless, such a metric does exist, when we start with an untwisted ω -slope-stable reflexive sheaf F of rank r and attempt to deform it as a twisted sheaf.

Definition 6.6. Let F be a reflexive ω -slope-stable (untwisted) sheaf of positive rank over (X, ω) . We say that F is *projectively ω -stable-hyperholomorphic* if, in addition, the sheaf $\mathcal{E}nd(F)$ is ω -polystable hyperholomorphic. A reflexive ω -slope-polystable sheaf is *projectively ω -polystable-hyperholomorphic*, if it is a direct sum of projectively ω -stable-hyperholomorphic sheaves.

Let F be a projectively ω -polystable-hyperholomorphic reflexive sheaf of rank $r > 0$. Let $\tilde{\theta} \in H^2(X, \mu_r)$ be the class $\exp(-2\pi\sqrt{-1}c_1(F)/r)$, as in equation (2.5). Denote by θ_t the image of $\tilde{\theta}$ in $H_{an}^2(X_t, \mathcal{O}_{X_t}^*)$. Similarly, let θ be the image of $\tilde{\theta}$ in $H_{an}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$ via the composite homomorphism

$$(6.2) \quad H^2(X, \mu_r) \rightarrow H^2(\mathcal{X}, \mu_r) \rightarrow H_{an}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*),$$

where the left homomorphism is the pull-back via the projection $\mathcal{X} \rightarrow X$, associated to the differentiable trivialization of the twistor deformation.

Construction 6.7. The sheaf F corresponds to a reflexive sheaf \mathcal{A} of Azumaya algebras (Definition 2.3) with Brauer class θ over the twistor space \mathcal{X} . Following is the construction of such a family. The sheaf $\mathcal{E}nd(F)$ is ω -polystable-hyperholomorphic, by assumption. Hence $\mathcal{E}nd(F)$ extends to a reflexive sheaf \mathcal{A} over \mathcal{X} . The structure on $\mathcal{E}nd(F)$ of a reflexive sheaf of Azumaya algebras extends to one on \mathcal{A} , by Lemma 6.5 part (3). It remains to prove that the Brauer class of \mathcal{A} is θ . Now \mathcal{A} has rank r and thus determines a class α in $H^2(\mathcal{X}, \mu_r)$ (use the homomorphism (2.3) and the fact that the singular locus of \mathcal{A} has codimension ≥ 3 in \mathcal{X}). The class α restricts to the class $\tilde{\theta}$ in $H^2(X, \mu_r)$. Hence, it suffices to prove that the image of the composite homomorphism (6.2) is equal to the r -torsion subgroup. Now $H^2(\mathcal{X}, \mu_r)$ is isomorphic to $H^2(\mathbb{P}^1, \mu_r) \oplus H^2(X, \mu_r)$ and the image of the summand $H^2(\mathbb{P}^1, \mu_r)$ in $H_{an}^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$ is trivial, as it is already trivial in $H_{an}^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^*)$.

Lemma 6.8. *Let F be a reflexive projectively ω -polystable-hyperholomorphic sheaf. Let \mathcal{A} be the reflexive sheaf of Azumaya algebras over the twistor family \mathcal{X} associated to $\mathcal{E}nd(F)$ via Construction 6.7. If \mathcal{A}_t is an ω_t -slope-stable sheaf of Azumaya algebras over X_t for some t , then it is ω_t -slope-stable for every t .*

Proof. Assume that \mathcal{A}'_t is a saturated subsheaf of \mathcal{A}_t of maximal parabolic subalgebras, and \mathcal{A}'_t has ω_t -slope 0, for some $t \in \mathbb{P}_\omega$. Then \mathcal{A}'_t extends as an ω -polystable-hyperholomorphic subsheaf P of \mathcal{A} , with P_0 an ω -slope 0 subsheaf of $\mathcal{E}nd(F)$, by Lemma 6.5 part 6. The subsheaf $P_t = \mathcal{A}'_t$ is a sheaf of maximal parabolic subalgebras. Its extension is also a subsheaf of maximal parabolic subalgebras, by Lemma 6.5 part 7. In particular, if P_t is a non-zero proper subsheaf, then \mathcal{A}_t is not ω_t -slope-stable, for any t . \square

⁷See however Proposition 7.8 below, which implies the existence of a Hermite-Einstein metric on $\mathcal{E}nd(E)$, in the special case of a θ -twisted reflexive sheaf E , such that the order r of the class θ is equal to $\text{rank}(E)$.

Theorem 6.9. ([Ve4], Corollary 3.24) *Let F be an ω -slope-polystable reflexive sheaf on (X, ω) , of ω -slope 0, and I_t an induced complex structure such that $I_t \notin \{I, -I\}$. Then*

$$(6.3) \quad \int_X \kappa_2(F) \omega^{2n-2} \geq \left| \int_X \kappa_2(F) \omega_t^{2n-2} \right|,$$

and equality holds, if and only if each stable direct summand F' of F is ω -stable-hyperholomorphic. Furthermore, equality holds in (6.3) if $\kappa_2(F)$ is of Hodge-type $(1, 1)$, with respect to I_t , for all $t \in \mathbb{P}_\omega$.

Proof. When F is ω -slope-stable this is precisely ([Ve4], Corollary 3.24). I thank Misha Verbitsky for pointing out this statement and the fact that the statement holds also when F is ω -slope-polystable. Assume that $F = \bigoplus_{i=1}^N F_i$, where F_i is ω -slope-stable. Then $\kappa_2(F) = \sum_{i=1}^N \kappa_2(F_i)$. We get

$$\int_X \kappa_2(F) \omega^{2n-2} = \sum_{i=1}^N \int_X \kappa_2(F_i) \omega^{2n-2} \geq \sum_{i=1}^N \left| \int_X \kappa_2(F_i) \omega_t^{2n-2} \right| \geq \left| \int_X \kappa_2(F) \omega_t^{2n-2} \right|,$$

where the first inequality is by [Ve4], Corollary 3.24, and the second by the triangle inequality. Clearly, equality holds above, if and only if it holds for each F_i .

If $\kappa_2(F)$ is of Hodge-type $(1, 1)$ with respect to I_t , for all $t \in \mathbb{P}_\omega$, then equality holds in (6.3), by Claim 3.21 in [Ve4]. \square

The following generalization of Theorem 6.3 was explained to me by Misha Verbitsky.

Corollary 6.10. (1) *Let E be an ω -slope-stable (untwisted) reflexive sheaf and assume that $\kappa_2(E)$ is $\text{Mon}(X)$ -invariant. Then $\mathcal{E}nd(E)$ is ω -polystable-hyperholomorphic and E is projectively ω -stable-hyperholomorphic.*

(2) *Let A be an ω -slope-stable reflexive sheaf of Azumaya algebras of rank r (Definition 6.4). Assume that the underlying rank r^2 coherent sheaf A is ω -slope-polystable. Finally assume that $c_2(A)$ is $\text{Mon}(X)$ -invariant. Then A extends to a reflexive sheaf \mathcal{A} of Azumaya algebras over \mathcal{X} , flat over \mathbb{P}_ω^1 , and \mathcal{A}_t is ω_t -slope-stable, for all $t \in \mathbb{P}_\omega^1$.*

Proof. (1) It suffices to prove that $\mathcal{E}nd(E)$ is ω -polystable-hyperholomorphic. Apply Theorem 6.9 with $F := \mathcal{E}nd(E)$. Equality holds in (6.3), since $\kappa_2(E)$ is of Hodge-type $(1, 1)$, for all $t \in \mathbb{P}_\omega$, by Lemma 3.2.

(2) A is isomorphic to A^* as a coherent sheaf, using the trace bilinear pairing, and thus $c_1(A) = 0$. The construction of \mathcal{A} follows from Theorem 6.9. The structure of Azumaya algebra extends, by Lemma 6.5 part (3). The stability of \mathcal{A}_t follows from Lemma 6.8. \square

6.3. Deformation of pairs along twistor paths. A *marking* of an irreducible holomorphic symplectic manifold X is an isometry $\phi : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ with a fixed lattice Λ . Let \mathfrak{M}_Λ be the moduli space of isomorphism classes of marked irreducible holomorphic symplectic manifolds [Hu]. A *twistor-path* in \mathfrak{M}_Λ is a sequence of twistor lines, in which each consecutive pair has non-trivial intersection in \mathfrak{M}_Λ . If the intersection, of each consecutive pair, contains a point corresponding to a manifold with trivial Picard group, we call the twistor-path *generic*.

Theorem 6.11. ([Ve3], Theorems 3.2 and 5.2.e) *Let (X_i, ϕ_i) , $i = 1, 2$, be two marked irreducible holomorphic symplectic manifolds, in the same connected component of \mathfrak{M}_Λ . Then there exists a generic twistor-path in \mathfrak{M}_Λ connecting (X_1, ϕ_1) with (X_2, ϕ_2) .*

We will need the following evident Lemma.

Lemma 6.12. *Let X be a compact Kähler manifold with a trivial Picard group $\text{Pic}(X) = \{\mathcal{O}_X\}$, ω and ω' two Kähler classes on X , and E a torsion free coherent \mathcal{O}_X -module of rank r . Then E is ω -slope-stable, if and only if E does not admit any subsheaf of rank r' , for $0 < r' < r$. In particular, E is ω -slope-stable, if and only if E is ω' -slope-stable.*

A parametrized twistor-path $\gamma : C \rightarrow \mathfrak{M}_\Lambda$ consists of a connected reduced nodal curve C , of arithmetic genus 0, with an ordering of the irreducible components, so that two consecutive components meet at a node, and a morphism γ from C to \mathfrak{M}_Λ , mapping the i -th component of C isomorphically onto a twistor line. If γ maps each node to a point with a trivial Picard group, we call γ a *generic parametrized twistor-path*. Let $\gamma : C \rightarrow \mathfrak{M}_\Lambda$ be a parametrized twistor-path, $\mathcal{X} \rightarrow C$ the natural twistor deformation, $0 \in C$ a point of the first component of C , and X_0 the fiber of \mathcal{X} over 0. Let E be a reflexive twisted sheaf on X_0 .

Definition 6.13. We say that E can be deformed along γ , if there exists a reflexive twisted coherent sheaf over \mathcal{X} , flat over C , which restriction to X_0 is isomorphic to E . Equivalently, there exists a reflexive sheaf of Azumaya $\mathcal{O}_\mathcal{X}$ -algebras, flat over C , which restriction to X_0 is isomorphic to $\mathcal{E}nd(E)$.

Let X be an irreducible holomorphic symplectic manifold and $\gamma : C \rightarrow \mathfrak{M}_\Lambda$ a generic parametrized twistor path, with $X_0 = X$. Let ω_0 be a Kähler class on X_0 , such that \mathbb{P}_{ω_0} is the first twistor line. Let $\omega_{t_i^-}$, $1 \leq i \leq N$, be a Kähler class on the i -th node X_{t_i} , such that $\mathbb{P}_{\omega_{t_i^-}}$ is the i -th twistor line, and $\omega_{t_i^+}$, $1 \leq i \leq N - 1$, a Kähler class on X_{t_i} , such that $\mathbb{P}_{\omega_{t_i^+}}$ is the $i + 1$ twistor line. Note that ω_0 determines $\omega_{t_1^-}$, and $\omega_{t_i^+}$ determines $\omega_{t_{i+1}^-}$. At a node $t_i \in C$, the group $\text{Pic}(X_{t_i})$ is trivial. Slope-stability is then independent of the Kähler class, by Lemma 6.12. We will abuse notation and say that a sheaf on X_{t_i} is ω_{t_i} -slope-stable, if it is slope-stable with respect to some, hence any Kähler class.

Proposition 6.14. (1) *Let F be an ω_0 -slope-stable (untwisted) reflexive sheaf. Assume that $\kappa_2(F)$ is $\text{Mon}(X)$ -invariant. Then F deforms along γ , in the sense of definition 6.13.*
 (2) *Let A be an ω_0 -slope-stable reflexive sheaf of Azumaya algebras of rank r (Definition 6.4). Assume that the underlying rank r^2 coherent sheaf A is ω -slope-polystable. Assume, furthermore, that $c_2(A)$ is $\text{Mon}(X)$ -invariant. Then A deforms along γ , as a reflexive sheaf of Azumaya algebras, in the sense of definition 6.13.*

Proof. (2) The following argument is similar to the proof of Theorem 10.8 in [Ve4]. The proof is by induction on the number N of twistor lines in C . A deforms along the first twistor line, by Corollary 6.10.

Assume that A deforms, as an ω_t -slope-stable sheaf of Azumaya algebras, along the first i twistor lines, and $i < N$. Then A_{t_i} is slope-polystable with respect to $\omega_{t_i^-}$ and hence also with respect to $\omega_{t_i^+}$, by Lemma 6.12. Hence, A_{t_i} is $\omega_{t_i^+}$ polystable-hyperholomorphic, by Theorem 6.9. The structure of an Azumaya algebra deforms along the $i + 1$ twistor line, by Lemma 6.5 part 3.

The ω_t -slope-stability of A_t is proven by induction as well. The stability for t in the first twistor line follows from Lemma 6.8. Stability of A_{t_1} for $\omega_{t_1^+}$ follows from that for $\omega_{t_1^-}$ and Lemma 6.12. The proof of the induction step is similar.

Part (1) follows from part (2), since $\mathcal{E}nd(F)$ is an ω_0 -slope-stable sheaf of Azumaya algebras and $c_2(\mathcal{E}nd(F))$ is a scalar multiple of $\kappa_2(F)$. \square

Remark 6.15. With the exception of Theorem 6.11, Verbitsky proves the results mentioned above for hyperkähler varieties, without assuming the condition $h^{2,0} = 1$ (the irreducibility condition). In particular, all the definitions and results in this section hold for $X \times X$, where X is an irreducible holomorphic symplectic manifold, provided the twistor deformations of $X \times X$ we consider are only fiber-square $\mathcal{X} \times_{\mathbb{P}_\omega^1} \mathcal{X}$ of twistor deformations of X , associated to a Kähler class ω on X .

7. STABLE HYPERHOLOMORPHIC SHEAVES OF RANK $2n - 2$ ON ALL MANIFOLDS OF $K3^{[n]}$ -TYPE

Definition 7.1. Let X be a complex manifold and E a torsion free θ -twisted coherent sheaf on X . E is said to be *very twisted*, if the rank of E is equal to the order of the class of θ in $H_{an}^2(X, \mathcal{O}_X^*)$.

A very-twisted sheaf does not have any non-trivial subsheaves of lower rank, so it is trivially slope-stable. In section 7.1 we construct a very twisted version of the sheaf E in Theorem 1.5. In section 7.2 we show that if F is very twisted, then the untwisted sheaf $\mathcal{E}nd(F)$ is ω -slope-polystable with respect to every Kähler class ω . Consequently, there exists a Hermite-Einstein metric on $\mathcal{E}nd(F)$. In section 7.3 we carefully degenerate the $K3$ surface, the moduli space \mathcal{M} , and the sheaf E to those in Theorem 1.5, so that the sheaf E is *untwisted*, yet still slope-stable. The stability Theorem 1.5 is proven in section 7.3. In section 7.4 we prove the deformability Theorem 1.6.

7.1. A very twisted $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$. We construct a very twisted reflexive sheaf $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$, over the self-product of a suitable choice of a moduli space \mathcal{M} (Theorem 7.4). Recall that an untwisted analogue of $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$ appears in Theorem 1.5.

Let $\mathcal{M}_H(v)$ be a smooth and projective moduli space of H -stable sheaves on a projective $K3$ surface S . Set $r := (v, v)$. Assume, that $(v, v) \geq 2$. Let μ_r be the group of r -th roots of unity.

Lemma 7.2. (1) *There exists a unique $r(v^\perp)$ coset \bar{w} in v^\perp of classes w , such that $w - v$ belongs to $rK_{top}S$.*

(2) *Define a class in $H^2(\mathcal{M}_H(v), \mu_r)$, by*

$$(7.1) \quad \tilde{\theta} := \exp(-2\pi\sqrt{-1}\bar{w}/r),$$

where we identify v^\perp with $H^2(\mathcal{M}_H(v), \mathbb{Z})$ via Mukai's isometry (3.3). Then the pair $\{\tilde{\theta}, \tilde{\theta}^{-1}\}$ is monodromy invariant.

Proof. 1) Uniqueness is clear. When v is the class of the ideal sheaf of a length n subscheme, with Mukai vector $(1, 0, 1 - n)$, choose $w = (1, 0, n - 1)$. The existence of such a class follows, for an arbitrary primitive class v with $(v, v) = 2n - 2$, since any two such classes belong to the same $O(K_{top}S)$ -orbit.

2) The class $\tilde{\theta}$ is determined by the primitive isometric lattice embedding $H^2(\mathcal{M}_H(v), \mathbb{Z}) \cong v^\perp \subset K_{top}S$ and the choice of a generator v of the line orthogonal to the image of $H^2(\mathcal{M}_H(v), \mathbb{Z})$. Any monodromy operator of $H^2(\mathcal{M}_H(v), \mathbb{Z})$ can be extended to an isometry of $K_{top}S$, which necessarily maps v to v or $-v$, by Theorem 5.2. \square

Let $\tilde{\theta}$ be the class in Equation (7.1). Denote by θ the image of $\tilde{\theta}$ in $H^2(\mathcal{M}_H(v), \mathcal{O}^*)$, via the sheaf inclusion $\iota : \mu_r \hookrightarrow \mathcal{O}^*$. Let $\beta : B \rightarrow \mathcal{M}_H(v) \times \mathcal{M}_H(v)$ be the blow-up of the diagonal and $\mathbb{P}V$ the projective bundle over B associated to the twisted locally free sheaf (4.4).

Lemma 7.3. (1) *The class $\tilde{\theta}(\mathbb{P}V) \in H^2(B, \mu_r)$, defined in (2.3), satisfies*

$$(7.2) \quad \tilde{\theta}(\mathbb{P}V) = \beta^* \left((\pi_1^* \tilde{\theta})^{-1} \pi_2^* \tilde{\theta} \right).$$

(2) *The order of the class θ in $H_{an}^2(\mathcal{M}_H(v), \mathcal{O}^*)$ is given by:*

$$\gcd\{(v, x) : x \in K_{top}S \text{ and } c_1(x) \text{ is of type } (1, 1)\}.$$

Proof. 1) Assume first, that v is the class of the ideal sheaf of a length n subscheme. Then V is a vector bundle, which restricts to the exceptional divisor D as a vector bundle with trivial determinant (Proposition 4.5). Thus, $c_1(V) = \beta^* \beta_* c_1(V) = \beta^* c_1(\mathcal{F})$, where \mathcal{F} is the object given in Equation (4.1). Now, $c_1(\mathcal{F}) = -\pi_1^* c_1(e_v) + \pi_2^* c_1(e_v)$, by Lemma 4.3. When \mathcal{E} is the universal ideal sheaf over $S \times S^{[n]}$, then $e_v = e_w$, where v has Mukai vector $(1, 0, 1-n)$, and that of w is $(1, 0, n-1)$, by Lemma 5.9 in [Ma4]. The coset \bar{w} in equation (7.1) is $w + (2n-2)K_{top}S$, since $w - v = (2n-2)(0, 0, 1)$. The equality (7.2) follows from equation (2.5).

The general case of equation (7.2) follows, by deformation of the classes on both sides, via a deformation to the Hilbert scheme case, as in Lemma 4.9.

2) Set $\mathcal{M} := \mathcal{M}_H(v)$. Consider the short exact sequence

$$(7.3) \quad 0 \rightarrow \mu_r \xrightarrow{\iota} \mathcal{O}^* \xrightarrow{(\bullet)^r} \mathcal{O}^* \rightarrow 0.$$

The connecting homomorphism $H^1(\mathcal{M}, \mathcal{O}^*) \rightarrow H^2(\mathcal{M}, \mu_r)$ sends the class of a line bundle L to $\exp(2\pi\sqrt{-1}c_1(L)/r)$. Let d be a positive integer dividing (v, v) . Then $\iota(d\tilde{\theta}) = 1$, if and only if $d\tilde{\theta} = \exp(-2\pi\sqrt{-1}\ell/r)$, for some $\ell \in H^{1,1}(\mathcal{M}, \mathbb{Z})$. Identify $H^2(\mathcal{M}, \mathbb{Z})$ with v^\perp , via Mukai's Hodge-isometry (3.3). Set $\bar{\ell} := \ell + rv^\perp$. It suffices to prove that the following are equivalent.

(1) There exists $\ell \in v^\perp$, with $c_1(\ell)$ of type $(1, 1)$, such that $\bar{\ell} = d\bar{w}$ in v^\perp/rv^\perp , where \bar{w} is the coset in Lemma 7.2.

(2) $d = (v, x)$, for some $x \in K_{top}S$, with $c_1(x)$ of type $(1, 1)$.

1 \Rightarrow 2: The $(1, 1)$ class $x := \frac{dv - \ell}{r}$ is integral, by the assumption on ℓ , and satisfies $(x, v) = \frac{d(v, v)}{r} = d$.

2 \Rightarrow 1: Set $\ell := dv - (v, v)x$. Then $(\ell, v) = 0$ and $\ell - dv = -rx$ belongs to $rK_{top}S$. Thus, $\bar{\ell} = d\bar{w}$ in v^\perp/rv^\perp , by Lemma 7.2 part 1. \square

Set $r := 2n-2$, $n \geq 2$. Let S be a projective $K3$ surface with a cyclic Picard group generated by an ample line-bundle H with $c_1(H)^2 = 2r^2 + r$. Let $v \in K_{top}S$ be the rank r class with $c_1(v) = c_1(H)$, and $\chi(v) = 2r$. Its Mukai vector $ch(v)\sqrt{td_S}$ is (r, H, r) . Then $(v, v) = r$ and $(v, x) \equiv 0$, (modulo r), for every class $x \in K_{top}S$ with $c_1(x)$ of type $(1, 1)$. The moduli space $\mathcal{M}_H(v)$ is smooth and projective (see section 3). Let E be the rank r $(\pi_1^*[\theta]^{-1}\pi_2^*[\theta])$ -twisted sheaf $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$, over $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$. E is reflexive, by Proposition 4.5.

Theorem 7.4. *The $(\pi_1^*[\theta]^{-1}\pi_2^*[\theta])$ -twisted sheaf E is ω -slope-stable (Definition 6.4) and the untwisted sheaf $\mathcal{E}nd(E)$ is ω -polystable-hyperholomorphic (Definition 6.1), with respect to every Kähler class ω on $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$.*

Proof. The class θ has order r , by lemma 7.3. It follows that E does not have any non-zero twisted subsheaves of rank $< r$ (see Remark 2.2). The class $\kappa_2(E)$ is monodromy-invariant, by Proposition 4.2. The polystability of $\mathcal{E}nd(E)$ is proven in the next section (Proposition 7.8). Consequently, $\mathcal{E}nd(E)$ is ω -polystable-hyperholomorphic, by Theorem 6.9. \square

7.2. Polystability of $\mathcal{E}nd(E)$ for a very twisted sheaf E .

Definition 7.5. Let X be a complex manifold and E a torsion free θ -twisted coherent sheaf on X . A subsheaf $A \subset \mathcal{E}nd(E)$ is said to be *degenerate*, if the generic rank of every local section of A is lower than that of E .

Lemma 7.6. *Let (X, ω) be a compact Kähler manifold, $\theta \in H_{an}^2(X, \mathcal{O}^*)$ a class of order $r > 0$, and E a reflexive, rank r , θ -twisted sheaf. Then $\mathcal{E}nd(E)$ does not have any non-zero degenerate subsheaf.*

Proof. The proof is by contradiction. Let A' be a non-zero degenerate subsheaf of $\mathcal{E}nd(E)$ and A its saturation in $\mathcal{E}nd(E)$. Then A is a reflexive degenerate subsheaf of $\mathcal{E}nd(E)$. Let $U \subset X$ be the open subset, where A is locally free, and set $Z := X \setminus U$. Then the codimension of Z in X is ≥ 3 .

We have the commutative diagram of exponential sequences

$$\begin{array}{ccccccc} H^2(X, \mathbb{Z}) & \rightarrow & H_{an}^2(X, \mathcal{O}) & \rightarrow & H_{an}^2(X, \mathcal{O}^*) & \rightarrow & H^3(X, \mathbb{Z}) \\ \cong \downarrow & & \rho_1 \downarrow & & \rho_2 \downarrow & & \downarrow \cong \\ H^2(U, \mathbb{Z}) & \rightarrow & H_{an}^2(U, \mathcal{O}) & \rightarrow & H_{an}^2(U, \mathcal{O}^*) & \rightarrow & H^3(U, \mathbb{Z}) \end{array}$$

Set $n := \dim_{\mathbb{C}}(X)$. The left and right vertical homomorphisms are isomorphisms, by the codimension of Z , Lefschetz Duality $H^i(U, \mathbb{Z}) \cong H_{2n-i}(X, Z, \mathbb{Z})$, and the vanishing of $H_{2n-i}(Z, \mathbb{Z})$, for $i < 6$. The homomorphism ρ_1 is injective, since the codimension of Z is ≥ 3 [Sch]⁸. It follows that the homomorphism ρ_2 is injective as well, by a diagram chase. We conclude, that the image θ' of θ in $H_{an}^2(U, \mathcal{O}^*)$ has order r .

Set $Y := \mathbb{P}[A|_U]$ and let $\pi : Y \rightarrow U$ be the natural morphism. The pull-back $\pi^* : H^2(U, \mathcal{O}^*) \rightarrow H^2(Y, \mathcal{O}^*)$ is injective, by a similar diagram chase, since the homomorphism $H^3(U, \mathbb{Z}) \rightarrow H^3(Y, \mathbb{Z})$ is injective, and both

$$H^2(U, \mathbb{Z})/c_1[\text{Pic}(U)] \rightarrow H^2(Y, \mathbb{Z})/c_1[\text{Pic}(Y)]$$

and $H_{an}^2(U, \mathcal{O}) \rightarrow H_{an}^2(Y, \mathcal{O})$ are isomorphisms. Hence, the pull-back $\theta'' := \pi^*(\theta')$ to Y has order r . Consequently, the θ'' -twisted sheaf π^*E does not have any non-trivial proper θ'' -twisted subsheaf. Let $\tau \subset \pi^*A$ be the tautological line subbundle. The image of the composition

$$\tau \otimes \pi^*E \rightarrow \pi^*(A \otimes E) \rightarrow \pi^*([\mathcal{E}nd(E)] \otimes E) \rightarrow \pi^*E$$

is a non-trivial θ'' -twisted proper subsheaf, since τ is a degenerate subsheaf of $\pi^*\mathcal{E}nd(E)$. A contradiction. \square

Lemma 7.7. *Let (X, ω) be a compact Kähler manifold, $\theta \in H_{an}^2(X, \mathcal{O}^*)$ a class of order $r > 0$, and E a reflexive, rank r , θ -twisted sheaf. Then $\mathcal{E}nd(E)$ is an ω -slope-semistable sheaf.*

⁸ If X is projective, and we consider the Zariski topology instead, the injectivity of ρ_1 follows from the vanishing of the cohomology $H_Z^i(X, \mathcal{O}_X)$, with support along Z , for $i \leq 2$ [H].

Proof. The proof is by contradiction. Assume that $\mathcal{E}nd(E)$ is not semi-stable, and let F be an ω -slope-stable destabilizing subsheaf of $\mathcal{E}nd(E)$ of maximal slope. Then $(F \otimes F)/\text{tor}$ is ω -slope-polystable of slope $2\mu(F)$. The image of $F \otimes F$ in $\mathcal{E}nd(E)$ must be zero, since otherwise the slope of the image is $\geq 2\mu(F)$, contradicting the assumption that the slope of F is maximal. We conclude that F is a degenerate subsheaf. We obtain a contradiction, by Lemma 7.6. \square

Proposition 7.8. *Let (X, ω) be a compact Kähler manifold, $\theta \in H_{an}^2(X, \mathcal{O}^*)$ a class of order $r > 0$, and E a reflexive, rank r , θ -twisted sheaf. Then $\mathcal{E}nd(E)$ is ω -slope polystable.*

Proof. $\mathcal{E}nd(E)$ is ω -slope semistable, by Lemma 7.7. Let $F \subset \mathcal{E}nd(E)$ be the maximal polystable subsheaf ([HL], Lemma 1.5.5). Then F is reflexive, and is hence locally free away from a closed analytic subvariety Z of codimension ≥ 3 in X . Let $F^\perp \subset \mathcal{E}nd(E)$ be the subsheaf orthogonal to F with respect to the trace-pairing on $\mathcal{E}nd(E)$. Set $A := F \cap F^\perp$.

We show first that A must vanish. Note first that the multiplication homomorphism $m : \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \rightarrow \mathcal{E}nd(E)$ maps $F \otimes F$ onto a subsheaf of slope 0. We conclude that the image is slope-polystable, and is hence contained in F . Consequently, F is a sheaf of unital associative subalgebras of $\mathcal{E}nd(E)$. Let a be a local section of A . Then a^n is a section of F , for all $n \geq 0$. Thus, $\text{tr}(a^k) = \text{tr}(a^{k-1}a) = 0$, for all $k > 0$. It follows that a is nilpotent. Hence, the sheaf A is a degenerate subsheaf of $\mathcal{E}nd(E)$. We conclude that $A = 0$, by Lemma 7.6.

We may assume that A vanishes. Thus the homomorphism

$$\phi : F \oplus F^\perp \longrightarrow \mathcal{E}nd(E)$$

is injective. Its degeneracy divisor, in $X \setminus Z$, must be trivial, since F and F^\perp must both have ω -slope 0. We conclude that ϕ is an isomorphism, since both its domain and target are reflexive sheaves. If F^\perp does not vanish, then it contains a stable subsheaf of ω -slope 0, contradicting the maximality of F . Thus $F = \mathcal{E}nd(E)$. \square

7.3. Stability of an untwisted $\mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E})$. Let $\pi : \mathcal{M} \rightarrow C$ be a non-isotrivial family of irreducible holomorphic-symplectic manifolds over a (connected) Riemann surface C . Fix a positive integer m . Assume given a global non-zero section α of $H^0(C, R_{\pi*}^2 \mu_m)$. Denote by C_α the subset of C given by

$$C_\alpha := \{t \in C : \alpha \text{ maps to the trivial class in } H_{an}^2(M_t, \mathcal{O}_{M_t}^*)\}.$$

Proposition 7.9. [Ogu] *C_α is a dense subset in the classical topology of C . Furthermore, either $C_\alpha = C$, or C_α is enumerable.*

Proof. The class $\alpha_t \in H^2(M_t, \mu_m)$ maps to the trivial class in $H_{an}^2(M_t, \mathcal{O}_{M_t}^*)$, if and only if α_t belongs to the image of $\text{Pic}(M_t)$ under the homomorphism $\tilde{\theta} : \text{Pic}(M_t) \rightarrow H^2(M_t, \mu_m)$, given by $\tilde{\theta}(L) = \exp(2\pi\sqrt{-1}c_1(L)/m)$. Indeed, we have seen that $\tilde{\theta}$ is the connecting homomorphism of the short exact sequence (7.3).

We may assume that the local system $R_{\pi*}^2 \mu_m$ is trivial, as the statement is local in C . Given a global section κ of the local system, denote by κ_t the corresponding class in $H^2(M_t, \mu_m)$. For each $\kappa \in H^0(C, R_{\pi*}^2 \mu_m)$, let C_κ be the subset of the curve C , consisting of points $t \in C$, such that κ_t is the image of a non-trivial class in $\text{Pic}(M_t)$. Theorem 1.1 in [Ogu] shows that the finite union $D := \bigcup_{\kappa \in H^0(C, R_{\pi*}^2 \mu_m)} C_\kappa$ is either equal to C or dense and enumerable. Proposition 7.9 states that each C_κ has this property and is thus a slight generalization of Theorem 1.1 in [Ogu]. Oguiso's proof is easily seen to establish this generalization. \square

Next we prove slope-stability of untwisted reflexive sheaves in a family containing a very-twisted reflexive sheaf.

Set $r := 2n - 2$, $n \geq 2$. Let $\pi : \mathcal{S} \rightarrow C$ be a non-isotrivial smooth and projective family of $K3$ surfaces, admitting a π -ample line-bundle H on \mathcal{S} of fiber-wise degree $2r^2 + r$. Assume, further, that there exists a point $0 \in C$, such that $\text{Pic}(S_0)$ is generated by H_0 . There exists a projective morphism $p : \mathcal{M} \rightarrow C$, whose fiber M_t , $t \in C$, is isomorphic to the moduli space $\mathcal{M}_{H_t}(r, H_t, r)$ of H_t -semistable sheaves on S_t of class (r, H_t, r) , by [Sim]. The fiber M_0 of p is smooth and connected of $K3^{[n]}$ -type, by our assumption on $\text{Pic}(S_0)$. Hence, we may assume that p is smooth and all its fibers are of $K3^{[n]}$ -type, possibly after restricting to a Zariski dense open subset of C . Let $\mathcal{O}_{\mathcal{M}}(1)$ be a p -ample line-bundle on \mathcal{M} and denote by $\mathcal{O}_{M_t}(1)$ its restriction to M_t . Note that $c_1(\mathcal{O}_{\mathcal{M}}(1))$ maps to a section of $R_{p*}^2 \mathbb{Z}$, which is in the image via the homomorphism (3.3) of the trivial local system of Mukai vectors spanned by the Mukai vectors $(2r + 1, c_1(H), 0)$ and $(1, 0, -1)$, by our assumption on $\text{Pic}(S_0)$.

There exists a θ -twisted universal sheaf \mathcal{E} over $\mathcal{S} \times_C \mathcal{M}$, for some class $\theta \in H_{an}^2(\mathcal{M}, \mathcal{O}_{\mathcal{M}}^*)$. Let \mathcal{E}_t be the restriction of \mathcal{E} to a twisted universal sheaf over $S_t \times M_t$ and denote by $\theta_t \in H^2(M_t, \mathcal{O}_{M_t}^*)$ its Brauer class. Set

$$E_t := \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^* \mathcal{E}_t, \pi_{23}^* \mathcal{E}_t).$$

E_t is a rank $2n - 2$ reflexive sheaf on $M_t \times M_t$ (Proposition 4.5). Given a point $m \in M_t$, denote by $E_{t,m}$ the restriction of E_t to $M_t \times \{m\}$. Let $ZD^{\mu s} \subset \mathcal{M}$ be the subset consisting points m , such that $E_{t,m}$ is $\mathcal{O}_{M_t}(1)$ -slope-stable. Let p_i be the projection from $M_t \times M_t$ onto the i -th factor. Set $\mathcal{O}_{M_t \times M_t}(1) := p_1^* \mathcal{O}_{M_t}(1) \otimes p_2^* \mathcal{O}_{M_t}(1)$. Let $\Sigma \subset C$ be the subset given by

$$\Sigma := \{t \in C : E_t \text{ is } \mathcal{O}_{M_t \times M_t}(1)\text{-slope-stable and } \theta_t \text{ is trivial}\}.$$

Theorem 7.10. *The subset Σ is a dense countable subset of C . The intersection of Σ with the image of $ZD^{\mu s}$ is a dense countable subset of C as well.*

Proof. Let $f_i : \mathcal{M} \times_C \mathcal{M} \rightarrow \mathcal{M}$, $i = 1, 2$, be the projections. Let ϕ_{ij} be the projection from $\mathcal{M} \times_C \mathcal{S} \times_C \mathcal{M}$ onto the fiber product of the i -th and j -th factors. The sheaf $\mathcal{E}xt_{\phi_{13}}^1(\phi_{12}^* \mathcal{E}, \phi_{23}^* \mathcal{E})$ is $(f_1^* \theta^{-1}) \cdot (f_2^* \theta)$ -twisted of rank r . Hence, the order of the class θ divides r . Consequently, the class θ lifts to a class $\tilde{\theta} \in H^2(\mathcal{M}, \mu_r)$. Let $C_{\theta} \subset C$ be the subset consisting of points $t \in C$, such that θ_t is trivial. We conclude that C_{θ} is countable and dense, by Proposition 7.9. Let C^s be the subset of C , consisting of points t where E_t is $\mathcal{O}_{M_t \times M_t}(1)$ -slope-stable. C^s contains the point 0, by Theorem 7.4. C^s is a Zariski open subset of C , by [Li], Corollary 2.3.2.11. Σ is the intersection $C^s \cap C_{\theta}$, which is a dense and countable subset of C . Lieblich's result also establishes that $ZD^{\mu s}$ is a Zariski open subset of \mathcal{M} . $ZD^{\mu s}$ contains the whole fiber M_0 , by Theorem 7.4. Hence, $ZD^{\mu s}$ is a Zariski dense open subset of \mathcal{M} . In particular, the image of $ZD^{\mu s}$ in C contains a Zariski dense open subset of C . \square

7.4. Proof of the deformability Theorem 1.6. Let n be an integer ≥ 2 . Set $r := 2n - 2$. Let S be a $K3$ surface admitting an ample line bundle H of degree $2r^2 + r$. Set $v := (r, H, r)$ and $\mathcal{M} := \mathcal{M}_H(v)$. Assume that \mathcal{M} is smooth, there exists an untwisted universal sheaf \mathcal{E} over $S \times \mathcal{M}$, and the reflexive sheaf

$$(7.4) \quad E := \mathcal{E}xt_{\pi_{13}}^1(\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E})$$

over $\mathcal{M} \times \mathcal{M}$ is $\mathcal{O}_{\mathcal{M} \times \mathcal{M}}(1)$ -slope-stable, where $\mathcal{O}_{\mathcal{M} \times \mathcal{M}}(1) := p_1^* \mathcal{O}_{\mathcal{M}}(1) \otimes p_2^* \mathcal{O}_{\mathcal{M}}(1)$ and $\mathcal{O}_{\mathcal{M}}(1)$ corresponds to a Mukai vector

$$a(1, 0, -1) + b(2r + 1, H, 0) \in v^\perp$$

via the Mukai isomorphism (3.3), for some integers a and b . The existence of such a pair (S, H) is proven in Theorem 7.10.

Theorem 7.11. *Let E be the sheaf given in equation (7.4) and X an irreducible holomorphic symplectic manifold of $K3^{[n]}$ -type. Then there exists a parametrized twistor path connecting $\mathcal{M}_H(v)$ and X , along which E can be deformed (in the sense of Definition 6.13).*

Proof. The proof is by induction on the length of the twistor path. The class $\kappa_2(E)$ is $\text{Mon}(\mathcal{M}_H(v))$ -invariant, by Proposition 4.2. Let ω' be the Kähler class of $\mathcal{O}_{\mathcal{M}_H(v) \times \mathcal{M}_H(v)}(1)$. Then E was chosen to be ω' -slope-stable. Hence, E is ω -slope-stable, for ω in an open neighborhood of ω' in the Kähler cone [HL], Ch. 4C. The sheaf E is projectively ω -stable-hyperholomorphic, by Corollary 6.10 and Remark 6.15. We choose ω , so that the hyperplane ω^\perp intersects trivially the lattice $H^{1,1}(\mathcal{M}_H(v), \mathbb{Z})$. Then $\text{Pic}(X_{t_1})$ is trivial, for a generic $t_1 \in \mathbb{P}_\omega^1$, by ([Hu], paragraph 1.17 page 76). There exists a generic parametrized twistor path from X_{t_1} to X , by Theorem 6.11. We get a generic parametrized twistor path from $\mathcal{M}_H(v)$ to X . We conclude that E deforms along the twistor path γ , by Proposition 6.14. \square

8. PROOF OF LEMMA 1.4

It suffices to prove the Lemma for every smooth and compact moduli space $\mathcal{M} := \mathcal{M}_H(v)$, for all $(v, v) \geq 2$. Let

$$\begin{aligned} u : K_{\text{top}} S &\longrightarrow H^*(\mathcal{M}, \mathbb{Q}) \\ u(x) &:= ch(e_x) \cdot \exp\left(\frac{-c_1(e_v)}{(v, v)}\right), \end{aligned}$$

where e_x is given in (3.1), and $u_{2i} : K_{\text{top}} S \rightarrow H^{2i}(\mathcal{M}, \mathbb{Q})$ the composition of u with the projection on the degree $2i$ -summand. Note that $u(v) = \kappa(e_v)$, $u_0(x) = (v, x)$,

$$u_2(x) = c_1(e_x) - \frac{(v, x)}{(v, v)} c_1(e_v),$$

$u_2(v) = 0$, and u_2 restricts to v^\perp and the standard Mukai isomorphism

$$(u_2)|_{v^\perp} : v^\perp \xrightarrow{\cong} H^2(\mathcal{M}, \mathbb{Z}).$$

Moreover, u is $O^+(K_{\text{top}} S)_v$ equivariant, by Theorem 5.1 and equation (5.5).

Let $\tilde{q} \in \text{Sym}^2 K_{\text{top}} S$ be the Mukai pairing. The following equality is a special case of equation (4.8) in [Ma2]:

$$(8.1) \quad c_2(T\mathcal{M}) = (u_2 \cup u_2 - 2u_4 \cup u_0)(\tilde{q}),$$

where $(u_2 \cup u_2 - 2u_4 \cup u_0)$ is the homomorphism from $K_{\text{top}} S \otimes K_{\text{top}} S$ to $H^4(\mathcal{M}, \mathbb{Q})$.

The orthogonal decomposition $(K_{\text{top}} S)_\mathbb{Q} = \mathbb{Q}v + (v^\perp)_\mathbb{Q}$ induces the decomposition $\tilde{q} = \frac{v \otimes v}{(v, v)} + q^{-1}$, where we identified v^\perp with $H^2(\mathcal{M}, \mathbb{Z})$, via u_2 . Equation (1.2) follows from (8.1) and the

following equations

$$(8.2) \quad (u_4 \cup u_0)(q^{-1}) = 0,$$

$$(8.3) \quad (u_2 \cup u_2)(q^{-1}) = q^{-1},$$

$$(8.4) \quad (u_2 \cup u_2)(v \otimes v) = 0,$$

$$(8.5) \quad (u_4 \cup u_0) \left(\frac{v \otimes v}{(v, v)} \right) = u_4(v) = \kappa_2(X).$$

Proof of Equation (8.2): $u_4 \cup u_0$ is $O^+(K_{top}S)_v$ -equivariant, and thus sends the $O^+(K_{top}S)_v$ -invariant class q^{-1} in $(v^\perp \otimes v^\perp)_\mathbb{Q}$ to an $O^+(K_{top}S)_v$ -invariant class in $u_4(v^\perp)_\mathbb{Q}$. But the image $u_4(v^\perp)$ either vanishes, or is an irreducible $O(K_{top}S)_v$ -module isomorphic to v^\perp . Thus, any invariant class in $u_4(v^\perp)$ vanishes.

Equations (8.3) and (8.5) are clear and Equation (8.4) follows from the vanishing of $u_2(v)$, observed above.

It remains to calculate the dimension of $\text{span}\{q^{-1}, c_2(TX), \kappa_2(X)\}$. The homomorphism $\text{Sym}^2 H^2(S^{[n]}, \mathbb{Q}) \rightarrow H^4(S^{[n]}, \mathbb{Q})$ is known to be injective [Ve1]. When $n = 2$, the homomorphism is surjective, by Göttsche's formula for the Betti numbers [Gö]. When $n = 3$, the co-kernel of the homomorphism is an irreducible 23-dimensional representation of $\text{Mon}(S^{[3]})$ [Ma2]. Thus, the monodromy invariant subspace of $H^4(X, \mathbb{Q})$ is one dimensional, and is spanned by each of the three classes, for X of $K3^{[n]}$ -type, $n \leq 3$.

Assume that $n \geq 4$. Then the monodromy invariant subspace of the quotient $H^4(S^{[n]}, \mathbb{Q})/\text{Sym}^2 H^2(S^{[n]}, \mathbb{Q})$ is one-dimensional and is spanned by the image of each of $\kappa_2(X)$ and $c_2(TX)$ ([Ma2], Lemma 4.9). \square

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